B.Sc. VI SEMESTER

Mathematics

PAPER – II

COMPLEX ANALYSIS
AND
RING THEORY

UNIT-IV

RESIDUE THEOREM, JORDAN’S LEMMA AND COUNTER INTEGRATION

Syllabus:
Unit – IV
Residue Theorem, Jordan’s Lemma and Contour Integration.
-10HRS

K.L.E Society's

G.I. Bagewadi Arts, Science & Commerce College, NIPANI

DEPARTMENT OF MATHEMATICS

B.Sc. VI Sem.Paper II

Lecture on poles and singularities

By Dr. M. M. Shankrikopp

Some more examples on poles:

Note: **If N^r is polynomial function or sinz or cosz or e^z and D^r is polynomial function then power of linear factors of D^r gives order of pole.**

Eg. Find poles of f(z) =$\frac{z}{(z-1)(z-2)^2}$

Clearly z=1 is pole of order 1 i.e simple pole and z=2 is a pole of order 2.

Eg. Find poles of f(z) =$\frac{ze^z}{(z-2)(z^2-5z+6)}$ = $\frac{ze^z}{(z-2[(z-2)(z-3)]}$ = $\frac{ze^z}{(z-2)^2(z-3)}$

Clearly z=3 is pole of order 1 i.e simple pole and z=2 is a pole of order 2.

Eg. Find poles of f(z) =$\frac{e^z}{z(z+4)^3(z-3)^4}$

Clearly z=0 simple pole, z=-4 is pole of order 3 and z=3 is a pole of order 4.

Eg. Find poles of f(z) =$\frac{z}{(z^2+z+1)}$ = $\frac{z}{(z-\alpha)(z-\beta)}$ **where** $\alpha = \frac{-1+\sqrt{1-4}}{2} = \frac{-1+i\sqrt{3}}{2}$

& $\beta = \frac{-1-i\sqrt{3}}{2}$

Clearly z= α, β are simple poles

Theorem: Zeros of analytic function are isolated

(i.e if z=a is zero of f(z) then it has no other zeros other than a in the nhd. of z=a)

Proof: Let f(z) be analytic and z=a be zero of the function f(z) of order m then by the definition f(z) = $(z-a)^m$ ($\emptyset(z)$) where $\emptyset(a) \neq 0$.

i.e $\emptyset(z)$ is analytic and non zero in the neighbourhood of z=a. Also $(z-a)^m \neq 0$ for all values $z \neq a$

Thus there exists no other points in the neighbourhood of z=a at which f(z) =0

Hence the zero z=a is isolated. It is true for all zeros of f(z).

$\therefore$ zeros of f(z) are isolated.

Theorem: poles of function are isolated.

(i.e if z=a is pole of f(z) then it has no other poles other than a in the nhd. of z=a)

Proof: Let z=a be a pole of order m of f(z), then by definition of pole principal part of f(z) in the Laurent's expansion have m no. of terms.

i.e $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \frac{b_3}{(z-a)^3} + \text{---------------------} \frac{b_m}{(z-a)^m}$

i.e $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_m}{(z-a)^m} + \frac{b_{m-1}}{(z-a)^{m-1}} + ------ \frac{b_2}{(z-a)^2} + \frac{b_1}{(z-a)}$ (reverse order)

i.e $f(z) = \frac{1}{(z-a)^m} \left[\sum_{n=0}^{\infty} a_n (z-a)^{n+m} + b_m + b_{m-1}(z-a) + b_{m-2}(z-a)^2 + \text{--------------} + b_1(z-a)^{m-1}\right]$

$= \frac{1}{(z-a)^m} \emptyset(z)$ ------------------(1)

where $\emptyset(z) = \left[\left(\sum_{n=0}^{\infty} a_n (z-a)^{n+m}\right) + b_m + b_{m-1}(z-a) + b_{m-2}(z-a)^2 + \text{---------} + b_1(z-a)^{m-1}\right]$

Clearly $\emptyset(z)$ does not tend to infinity for any finite value of z as powers of (z-a) are positive.

⇨ There is no other pole in the nhd. of z=a.

$\therefore$ from (1), f(z) has only the pole z=a and no other poles in the nhd. of z=a.

Thus poles of f(z) are isolated.

Note: Both theorems are important

Unit IV

Definition of residue, Cauchy's Residue Theorem and Counter Integration

Definition of Residue (important for 2 marks): Let f(z) be analytic and z=a be a pole of f(z) of order m inside closed curve C then by Lauernt's Theorem we have

f(z) = $\sum_{n=0}^{\infty} a_n(z-a)^n$ + $b_1 \frac{1}{(z-a)}$ + $b_2 \frac{1}{(z-a)^2}$ + ----------+ $b_m \frac{1}{(z-a)^m}$ where

$b_m = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^{-m+1}}dz$

Here particularly b_1 = $\frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^{-1+1}}\boldsymbol{dz} = \frac{1}{2\pi i}\int_c \boldsymbol{f(z)\,dz}$ is called residue of the function f(z) at a pole z=a.(i.e coefficient of $\frac{1}{(z-a)}$ in the Prin. Part, i.e the term left after + ve power)

Note: Residues are usually denoted by R_1 , R_2, -------

For example: If f(z)= $\frac{z}{(z-1)(z-3)^2}$, clearly z=1 is a pole of order 1 (simple pole) and z=3 is a pole of order 2

By using partial fraction we have f(z) = $\frac{1}{4(z-1)}$ - $\frac{1}{4(z-3)}$ + $\frac{3}{2(z-3)^2}$, here coefficient of $\frac{1}{(z-1)}$ is $\frac{1}{4}$ and coefficient of $\frac{1}{(z-3)}$ is $\frac{-1}{4}$

∴ residue of f(z) at z=1 is $\mathbf{R_1} = \frac{1}{4}$ and residue of f(z) at z=3 is $\mathbf{R_2} = -\frac{1}{4}$

Calculation of residues:

Calculation of residue by using above partial fraction method is tedious if more factors are there in D^r. So there are easy methods to calculate residues.

(i) **Calculation of residue of f(z) at simple pole (pole of order 1)**: If z =a is a pole of f(z) of order 1 then by Laurent's Theorem we have

f(z) = $\sum_{n=0}^{\infty} a_n(z-a)^n$ + $b_1 \frac{1}{(z-a)}$

Multiplying throughout by (z-a) , we get

(z-a) f(z) = $\sum_{n=0}^{\infty} a_n(z-a)^{n+1}$ + b_1

Taking the limit as z→a on both the sides we get

$\lim_{z\to a}(z-a)f(z) = \lim_{z\to a}\sum_{n=0}^{\infty} a_n(z-a)^{n+1}$ + b_1

i.e $\lim_{z\to a}(z-a)f(z)$ = 0 + b_1 => $b_1 = \lim_{z\to a}(z-a)f(z)$

Thus if z =a is a pole of f(z) of order 1(simple pole) then residue of f(z) is obtained by $R_1 = \lim_{z\to a}(z-a)f(z)$

(ii) Calculation of residue of f(z) at pole of order m :

Above method is not applicable if z=a is pole of order more than 1

Thm. (Important for 5 marks) : Prove that z=a be a pole of f(z) of order m then residue of f(z) at z=a is $R_1 = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}((z-a)^m f(z))$.

Proof: Let z=a be a pole of f(z) of order m , then

$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + b_1\frac{1}{(z-a)} + b_2\frac{1}{(z-a)^2} + \text{----------} + b_m\frac{1}{(z-a)^m}$

$= \frac{1}{(z-a)^m}[\sum_{n=0}^{\infty} a_n(z-a)^{n+m} + b_1(z-a)^{m-1} + b_2(z-a)^{m-2} + \text{------ } b_m]$

$= \frac{1}{(z-a)^m}\emptyset(z)$

where $\emptyset(z) = \sum_{n=0}^{\infty} a_n(z-a)^{n+m} + b_1(z-a)^{m-1} + b_2(z-a)^{m-2} + \text{------ } b_m$

$\therefore f(z) = \frac{\emptyset(z)}{(z-a)^m}$ ---------------------------(1) where $\emptyset(z)$ is analytic at z=a.

By the definition of residue of f(z) we have $b_1 = \frac{1}{2\pi i}\int_c f(z)\,dz$

$= \frac{1}{2\pi i}\int_c \frac{\emptyset(z)}{(z-a)^m}dz = \frac{1}{(m-1)!}\emptyset^{m-1}(a)$ by C. I. formula for n^{th} derivative .

$= \frac{1}{(m-1)!}\lim_{z\to a}\emptyset^{m-1}(z) = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(\emptyset(z))$

$= \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)$

From (1)

Thus residue of f(z) at pole z=a of order m is

$R_1 = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}((z-a)^m f(z))$.

Note: Sometimes residue of f(z) at pole z=a is also written as Res.(f,a)

Examples on calculation of residues:

1. Find the residue of f(z) $= \frac{z}{z^2-1}$ at z=1 (2017)

Soln.: Now $f(z) = \frac{z}{z^2-1} = \frac{z}{(z-1)(z+1)}$

Clearly z =1 and z=-1 are poles of order 1, ie simple poles

If R_1 is residue of f(z) at z=1 then $R_1 = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1)f(z)$

$= \lim_{z\to 1}\cancel{(z-1)}\frac{z}{\cancel{(z-1)}(z+1)} = \frac{1}{2}$

$\therefore R_1 = \frac{1}{2}$ is a residue at 1.

2. Find the residue of f(z) = $\frac{e^z}{(z^2+1)^2}$ at z=i **(2017)**

Soln.: Now f(z) = $\frac{e^z}{(z^2+1)^2}$ = $\frac{e^z}{(z+i)^2(z-i)^2}$

Clearly z =i and z= -i are poles of order 2

If R_1 is residue of f(z) at z=i then $R_1 = \frac{1}{(2-1)!}\lim_{z\to i}\frac{d}{dz}(z-i)^2 f(z)$

$$= \lim_{z\to i}\frac{d}{dz}(z-i)^2\frac{e^z}{(z+i)^2(z-i)^2}$$

$$= \lim_{z\to i}\frac{d}{dz}\frac{e^z}{(z+i)^2} = \lim_{z\to i}\frac{(z+i)^2e^z-2(z-i)e^z}{(z+i)^4} = \lim_{z\to i}\frac{(z+i)e^z-2e^z}{(z+i)^3}$$

∴ $R_1 = \frac{2(i-1)e^i}{-8i} = \frac{i(i-1)e^i}{4}$ is a residue at i.

3. Find the residue of f(z) = $\frac{z^4}{z^2+a^2}$ at all its poles. **(2016)**

Soln.: Now f(z) = $\frac{z^4}{z^2+a^2}$ = $\frac{z^4}{(z-ai)(z+ai)}$

Clearly z =ai and z=-ai are poles of order 1, ie simple poles

If R_1 is residue of f(z) at z=ai then $R_1 = \lim_{z\to ai}(z-ai)\, f(z) = \lim_{z\to ai}(z-ai)\, f(z)$

$$= \lim_{z\to ai}\cancel{(z-ai)}\frac{z^4}{\cancel{(z-ai)}(z+ai)} = \frac{a^3}{2i}$$

∴ $R_1 = \frac{a^3}{2i}$ is a residue at z = ai.

If R_2 is residue of f(z) at z= - ai then $R_2 = \lim_{z\to -ai}(z+ai)\, f(z) = \lim_{z\to -ai}(z+ai)\, f(z)$

$$= \lim_{z\to -ai}\cancel{(z+ai)}\frac{z^4}{(z-ai)\cancel{(z+ai)}} = \frac{a^3}{-2i}$$

∴ $R_2 = -\frac{a^3}{2i}$ is a residue at z = - ai.

HOME work

5. Find the residue of f(z) = $\frac{2z+3}{(z-1)(z-2)}$ at z=2 **(2016)**

6. Find the residue of f(z) = $\frac{z}{z^2+1}$ at its all poles **(2015)**

7. Find the residue of f(z) = $\frac{z}{(z-1)(z-2)}$ at z=2 **(2014)**

8. Find the residues of f(z) = z/(z^2+1) at its poles **(2015)**

HOME work

9. Find the residues of f(z) = $\frac{2z+3}{(z-1)(z-2)}$ at z=2 **(2013, 2016)**

Soln.: Now f(z) = $\frac{2z+3}{(z-1)(z-2)}$ Clearly z =1 and z=2 are poles of order 1, ie simple poles

If R_1 is residue of f(z) at z=2 then $R_1 = \lim_{z\to 2}(z-2)\, f(z) = \lim_{z\to 2}(z-2)\frac{2z+3}{(z-1)(z-2)} = \frac{7}{1}$

∴ $R_1 = 7$ is a residue at z= 2.

10. Find the residues of f(z) = $\frac{e^z}{z(z-1)^2}$ at z=0 (2013)

Soln.: Now f(z) $= \frac{e^z}{z(z-1)^2}$ Clearly z =0 is simple pole and z=1 is pole of order 2.

If R_1 is residue of f(z) at z=0 then $R_1 = \lim_{z \to 0}(z-0)\, f(z) = \lim_{z \to 0}(z)\frac{e^z}{z(z-1)^2} = \frac{1}{1}$

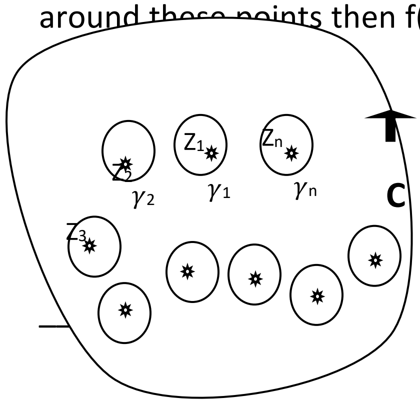
∴ $R_1 = 1$ is a residue at z= 0.

Cauchy's Residue Theorem (Compulsory question for 5 marks)

Statement: Let f(z) be analytic within and on closed contour C except at finite no. of poles $z_1, z_2, z_3,$ -----------z_n inside C then $_c\int f(z)dz$ dz = 2πi($R_1 + R_2 + R_3$ +------------------------R_n) = 2πi(sum of residues at these poles inside C)

where R_1, R_2, R_3------------------R_n are residues at poles $z_1, z_2, z_3,$ -----------z_n resply.

Proof: By hypothesis $z_1, z_2, z_3,$ -----------z_n poles of f(z) inside C. Therefore function f(z) is not analytic at these points in side C. Hence construct small circles $\gamma_1, \gamma_2, \gamma_3,$ --------------------------------γ_n around these points then f(z) is analytic in the egion bounded by closed curves C, $\gamma_1, \gamma_2, \gamma_3,$ - -------



By Cauchy's theorem for multi connected region we have

$$_c\int f(z)dz = {}_{\gamma_1}\int f(z)dz + {}_{\gamma_2}\int f(z)dz + \text{-------} {}_{\gamma_n}\int f(z)dz \quad \text{---(1)}$$

By definition of residue of f(z) we have

$R_1 = \frac{1}{2\pi i}\ {}_{\gamma_1}\int f(z)dz$ where γ_1 is circle around the pole z_1 and R_1 is res.

∴ ${}_{\gamma_1}\int f(z)dz = 2\pi i\, R_1$

Similarly ${}_{\gamma_2}\int f(z)dz = 2\pi i\, R_2$, ${}_{\gamma_3}\int f(z)dz = 2\pi i\, R_3$, ----------------${}_{\gamma_n}\int f(z)dz = 2\pi i\, R_n$

Then (1) becomes , $_c\int f(z)dz = 2\pi i\, R_1 + 2\pi i\, R_2 + 2\pi i\, R_3$ +------------------$2\pi i\, R_n$

$= 2\pi i\,(R_1 + R_2 + R_3$ +------------------$R_n)$

$= 2\pi$i (sum of residues at these poles inside C)

Thus if f(z) be analytic within and on closed contour C except at finite no. of poles $z_1, z_2, z_3,$ -----------z_n inside C then

$_c\int f(z)dz$ dz = 2πi($R_1 + R_2 + R_3$ +-------------- R_n) = 2πi(sum of residues at these poles inside C)

where R_1, R_2, R_3------------------R_n are residues at poles $z_1, z_2, z_3,$ -----------z_n resply.

Hence the proof

Evaluation of integrals using C.R. Theorem:

We are going to solve three types of examples using C.R. theorem

(i) $_c\int f(z)dz$ where C is closed curve

(ii) $\int_0^{2\pi} f(\sin\theta, \cos\theta)d\theta$

(iii) $\int_{-\infty}^{\infty} f(x)dx$ or $\int_0^{\infty} f(x)dx$

} Real integrals

Evaluation of Examples on type (i): $_c\int f(z)dz$ where C is closed curve

We already solved this type of examples by Cauchy's integral formula, but by usingC. R. theorem easily we evaluate.

Procedure: 1. Consider f(z), find poles and their orders.

2. See which poles are inside C

3. Calculate residues at these poles by calculation of residues method

4. Apply C. R. thm $_c\int f(z)dz$ dz = 2πi(sum of residues at these poles inside C)

1.Evaluate $_c\int \frac{sinz}{(z-\pi)^3}$ dz where C; lzl =4 **(2016)**

Soln.: Now given intergal $_c\int f(z)dz$ = $_c\int \frac{sinz}{(z-\pi)^3}$ dz where f(z) $=\frac{sinz}{(z-\pi)^3}$ Clearly z $=\pi$ is a pole of order 3 which is inside the circle lzl =4 .

If R_1 is residue of f(z) at z= π then $R_1 = \frac{1}{(3-1)!} \lim_{z\to\pi} \frac{d^2}{dz^2}(z-\pi)^3 f(z)$

$$=\frac{1}{2!} \lim_{z\to\pi} \frac{d^2}{dz^2}\cancel{(z-\pi)^3} \frac{sinz}{\cancel{(z-\pi)^3}}$$

$$= \lim_{z\to\pi} \frac{d^2}{dz^2} sinz = \lim_{z\to\pi} (-sinz) = 0$$

$$= 0$$

∴ By C. R. Thm $_c\int f(z)dz$ = $_c\int \frac{sinz}{(z-\pi)^3}$ dz = 2πi (R_1) = 2πi(0)= 0

2.Evaluate $_c\int$ dz over closed contour C. **(2014, 2015)**

Soln: Given integral $_c\int$ f(z) dz = $_c\int$ dz where f(z) =1 which is analytic every where.

∴ By Cauchy's Thm $_c\int$ dz = 0.

3. Evaluate $_c\int \frac{z}{z^2+2z-3}$ dz where C is lzl = 2 **(2017)**

Soln.: Now given intergal $_c\int f(z)dz$ = $_c\int \frac{z}{z^2+2z-3}$ dz where f(z) $=\frac{z}{z^2+2z-3} = \frac{z}{(z-1)(z+3)}$

Clearly z $=1$ and -3 are simple poles of f(z) for which z=1 is inside the circle lzl =2 .

∴We have to calculate residue only at z=1.

If R_1 is residue of f(z) at z=1 then $R_1 = \lim_{z\to1}(z-1) f(z) = \lim_{z\to1}(z-1) f(z)$

$$= \lim_{z\to1}\cancel{(z-1)} \frac{z}{\cancel{(z-1)}(z+3)} = \frac{1}{4}$$

∴ $R_1 =\frac{1}{4}$ is a residue at 1.

∴ By C. R. Thm $c\int f(z)dz = c\int \frac{z}{z^2+2z-3} dz = 2\pi i\,(R_1) = 2\pi i(\frac{1}{4}) = \frac{\pi i}{2}$

4. Prove that $_c\int \frac{e^z}{z^{n+1}} dz = \frac{2\pi i}{n!}$ where C is lzl = 2 .

Soln.: Now given integral $c\int f(z)dz = c\int \frac{e^z}{z^{n+1}} dz$ where $f(z) = \frac{e^z}{z^{n+1}}$ Clearly z =0 is a pole of order (n+1) which is inside the circle lzl =2 .

If R_1 is residue of f(z) at z= 0 then $R_1 = \frac{1}{(n+1-1)!} \lim_{z\to 0} \frac{d^n}{dz^n}(z-0)^n f(z)$

$$= \frac{1}{n!} \lim_{z\to 0} \frac{d^n}{dz^n}\cancel{(z)^n} \frac{e^z}{\cancel{(z)^n}}$$

$$= \frac{1}{n!} \lim_{z\to \pi} \frac{d^n}{dz^n} e^z = \frac{1}{n!} \lim_{z\to 0} e^z = \frac{1}{n!}$$

∴ By C. R. Thm $c\int f(z)dz = {}_c\int \frac{e^z}{z^{n+1}}$ **dz** $= 2\pi i\,(R_1) = 2\pi i(\frac{1}{n!}) = \frac{2\pi i}{n!}$

5. Evaluate $_c\int z^3/(z+1)\,dz$ if c is lzl = 2

Soln: Now given intergal $c\int f(z)dz = c\int \frac{z^3}{z+1} dz$ where $f(z) = \frac{z^3}{z+1}$

Clearly z =-1 *is* simple pole of f(z) which is inside the circle lzl =2 .

∴We have to calculate residue only at z= - 1.

If R_1 is residue of f(z) at z= -1 then $R_1 = \lim_{z\to -1}(z+1)\, f(z) = \lim_{z\to -1}\cancel{(z+1)} \frac{z^3}{\cancel{z+1}} = \underline{-1}$

=

∴ $R_1 = -1$ is a residue at -1.

∴ By C. R. Thm $c\int f(z)dz = c\int \frac{z^3}{z+1} dz = 2\pi i\,(R_1) = 2\pi i(-1) = -2\pi i$

6. Evaluate $_c\int dz/(z-2)$ if c is lz-2l = 4 (2016)

7. Evaluate $_c\int dz/z(z^2+4)$ if c is lzl = 1

Try 6 and 7 as exercise.

8. Obtain residues of $f(z) = \frac{\cos z}{z(z-1)^2}$ at all singularities and hence evaluate c∫f(z) dz where c is lzl = 2

Soln: Now given intergal $c\int f(z)dz = c\int \frac{\cos z}{z(z-1)^2} dz$ where $f(z) = \frac{\cos z}{z(z-1)^2}$ Clearly z =0 is a pole of order 1 and z =1 pole of order 2, both are inside the circle lzl =2 .

If R_1 is residue of f(z) at pole z= 0 then $R_1 = \lim_{z\to 0}(z)\, f(z) = \lim_{z\to 0}(z) \frac{\cos z}{z(z-1)^2} = \lim_{z\to 0} \frac{\cos z}{(z-1)^2} = 1$

∴ $R_1 = 1$

If R_2 is residue of f(z) at pole z= 1 then $R_2 = \frac{1}{(2-1)!} \lim_{z\to 1} \frac{d}{dz}(z-1)^2 f(z)$

$$= \frac{1}{2!} \lim_{z\to 1} \frac{d}{dz}(z-1)^2 \frac{\cos z}{z(z-1)^2}$$

$= \lim_{z\to 1} \frac{d}{dz}\left[\frac{\cos z}{z}\right] = \lim_{z\to 1}\left[\frac{z\,(-\sin z) - \cos z\,(1)}{z^2}\right] = -(\sin 1 + \cos 1)$

$R_2 = -(\sin 1 + \cos 1)$

∴ By C. R. Thm $c\int f(z)dz = c\int \frac{\cos z}{z(z-1)^2}\,dz = 2\pi i\,(R_1 + R_2) = 2\pi i(1 + \sin 1 + \cos 1).$

9.Evaluate c∫ $z/[(z^2+1)(z^2-9)]$ dz where c is the circle lzl = 2

Soln: Now given intergal $c\int f(z)dz = c\int \frac{z}{(z^2+1)(z^2-9)}\,dz$ where $f(z) = \frac{z}{(z^2+1)(z^2-9)} = \frac{z}{(z+i)(z-i)(z+3)(z-3)}$ and C is circle lzl=2

Clearly z $= i, -i, 3, -3$ simple poles for which z= $i, -i$ lie inside the circle C.

∴ we have to calculate residues only at these two poles.

If R_1 is residue of f(z) at pole z= i then $R_1 = \lim_{z\to i}(z-i)\,f(z)$

$= \lim_{z\to i}(z-i)\frac{z}{(z+i)(z-i)(z+3)(z-3)}$

$= \lim_{z\to i}\frac{z}{(z+i)(z^2-9)} = \frac{i}{(2i)(-10)} = \frac{1}{-20}$

∴ $R_1 = -\frac{1}{20}$

If R_2 is residue of f(z) at pole z= $-i$ then $R_2 = \lim_{z\to -i}(z+i)f(z)$

$= \lim_{z\to -i}(z+i)\frac{z}{(z+i)(z-i)(z+3)(z-3)}$

$= \lim_{z\to -i}\frac{z}{(z-i)(z^2-9)} = \frac{-i}{(-2i)(-10)} = -\frac{1}{20}$

∴ $R_2 = -\frac{1}{20}$

∴ By C. R. Thm $c\int f(z)dz = c\int \frac{z}{(z^2+1)(z^2-9)}\,dz = 2\pi i\,(R_1 + R_2) = 2\pi i\left(-\frac{1}{20} - \frac{1}{20}\right) = -\frac{\pi i}{5}$

10. Evaluate c∫ $dz/[z^2(z+4)]$ where c is the circle lzl = 5

Soln: Now given intergal $c\int f(z)dz = c\int \frac{1}{z^2(z+4)}\,dz$ where $f(z) = \frac{1}{z^2(z+4)}$ Clearly z =0 is a pole of order 2 and z =-4 is simple pole both are inside the circle lzl =5 .

If R_1 is residue of f(z) at pole z= 0 then $R_1 = \frac{1}{(2-1)!}\lim_{z\to 0}\frac{d}{dz}z^2\,f(z)$

$= \frac{1}{2!}\lim_{z\to 0}\frac{d}{dz}z^2\,\frac{1}{z^2(z+4)}$

$= \lim_{z\to 0}\frac{d}{dz}\left[\frac{1}{z+4}\right] = \lim_{z\to 0}\left[-\frac{1}{(z+4)^2}\right] = -\frac{1}{16}$

∴ $R_1 = -\frac{1}{16}$

If R_2 is a residue of f(z) at simple pole z= - 4 then $R_2 = \lim_{z\to -4} (z+4)\, f(z)$

$$= \lim_{z\to-4} (z+4) \frac{1}{z^2(z+4)}$$

$$= \lim_{z\to-4} \frac{1}{z^2} = \frac{1}{16}$$

$\therefore R_2 = \frac{1}{16}$

$\therefore$ By C. R. Thm $c\int f(z)dz = c\int \frac{1}{z^2(z+4)} dz = 2\pi i\,(R_1+R_2) = 2\pi i\left(\frac{-1}{16} + \frac{1}{16}\right) = 0$

11. Evaluate c∫ z dz/[(z+i)(9-z²)] dz where c is the circle lzl = 2 (2014)

Exercise

Soln:

12. Evaluate c∫ (2z+1)/(z²+z-6) dz where c is the circle lzl = 4 (2008)

Soln: Now given intergal $c\int f(z)dz = c\int \frac{2z+1}{z^2+z-6} dz$ where $f(z) = \frac{2z+1}{z^2+z-6} = \frac{2z+1}{(z+3)(z-2)}$ Clearly $z = -3$ *and* $z = 2$ *are* simple poles and both lie inside the circle lzl =4 .

If R_1 is residue of f(z) at pole $z = -3$ then $R_1 = \lim_{z\to -3} (z+3)\, f(z)$

$$= \lim_{z\to-3} (z+3) \frac{2z+1}{(z+3)(z-2)}$$

$$= \lim_{z\to-3} \frac{2z+1}{(z-2)} = \frac{-5}{-5} = 1$$

$\therefore R_1 = \mathbf{1}$

If R_2 is a residue of f(z) at simple pole z= 2 then $R_2 = \lim_{z\to 2}(z-2)\, f(z)$

$$= \lim_{z\to 2}(z-2)\frac{\mathbf{2z+1}}{\mathbf{(z+3)(z-2)}}$$

$$= \lim_{z\to 2}\frac{\mathbf{2z+1}}{\mathbf{(z+3)}} = \frac{\mathbf{5}}{\mathbf{5}} = 1$$

$\therefore R_2 = \mathbf{1}$

$\therefore$ By C. R. Thm $c\int f(z)dz = c\int\frac{\mathbf{2z+1}}{\mathbf{z^2+z-6}}dz = 2\pi i\,(R_1+R_2) = 2\pi i(\mathbf{1}+\mathbf{1}) = 4\pi i$

13. Evaluate c∫ (2z+1)/(z-2)(z+3)(z+1) dz where c is the circle lzl = 5/2

Soln: Now given intergal $c\int f(z)dz = c\int\frac{\mathbf{2z+1}}{\mathbf{(z-2)(z+3)(z+1)}}dz$ where $f(z) = \frac{\mathbf{2z+1}}{\mathbf{(z-2)(z+3)(z+1)}}$ and C is lzl = $\frac{5}{2}$ Clearly $z = -3$, $z = 2$ and $z = -1$ are simple poles for which z=2 and -1 lie inside the circle lzl =$\frac{5}{2}$.

If R_1 is residue of f(z) at pole $z = -1$ then $R_1 = \lim_{z\to -1}(z+1)\, f(z)$

$$= \lim_{z\to -1}(z+1)\frac{\mathbf{2z+1}}{\mathbf{(z+3)(z-2)(z+1)}}$$

$$R_1 = \lim_{z\to -1}\frac{\mathbf{2z+1}}{\mathbf{(z-2)(z+3)}} = \frac{\mathbf{-1}}{\mathbf{-6}} = \frac{\mathbf{1}}{\mathbf{6}}$$

If R_2 is a residue of f(z) at simple pole z= 2 then $R_2 = \lim_{z\to 2}(z-2)\, f(z)$

$$= \lim_{z\to 2}(z-2)\frac{\mathbf{2z+1}}{\mathbf{(z+3)(z-2)(z+1)}}$$

$$= \lim_{z\to 2}\frac{\mathbf{2z+1}}{\mathbf{(z+3(z+1))}} = \frac{\mathbf{5}}{\mathbf{15}} = \frac{\mathbf{1}}{\mathbf{5}}$$

$\therefore R_1 = \frac{\mathbf{1}}{\mathbf{6}}$ and $R_2 = \frac{\mathbf{1}}{\mathbf{5}}$

$\therefore$ By C. R. Thm $c\int f(z)dz = c\int\frac{\mathbf{2z+1}}{\mathbf{(z-2)(z+3)(z+1)}}dz = 2\pi i\,(R_1+R_2) = 2\pi i(\frac{\mathbf{1}}{\mathbf{6}}+\frac{\mathbf{1}}{\mathbf{5}}) = 2\pi i\,(\frac{\mathbf{11}}{\mathbf{30}}) = (\frac{\mathbf{11\pi i}}{\mathbf{15}})$

14. Evaluate (i) c∫ dz/[z(z²+4)] where c is the circle lzl = 5 (ii) c∫$\frac{\mathbf{z^2-4}}{\mathbf{z(z^2+9)}}$ dz where C; lzl =1 (2015)

Soln: (i) Now given intergal $c\int f(z)dz = c\int\frac{\mathbf{1}}{\mathbf{z(z^2+4)}}dz$ where $f(z) = \frac{\mathbf{1}}{\mathbf{z(z^2+4)}} = \frac{\mathbf{1}}{\mathbf{z(z+2i)(z-2i)}}$ and C is lzl = 5 Clearly $z = 0$, $z = 2i$ and $z = -2i$ are simple poles and all lie inside the circle lzl = 5 .

$\therefore$ we have to calculate residues at all these poles.

If R_1 is residue of f(z) at pole z= 0 then $R_1 = \lim_{z\to 0} z\, f(z) = \lim_{z\to 0} z\,\frac{\mathbf{1}}{\mathbf{z(z^2+4)}}$

$$= \lim_{z\to 0}\frac{\mathbf{1}}{\mathbf{(z^2+4)}} = \frac{\mathbf{1}}{\mathbf{4}}$$

$\therefore R_1 = \frac{\mathbf{1}}{\mathbf{4}}$

If R_2 is a residue of f(z) at simple pole z= 2i then $R_2 = \lim_{z\to 2i}(z-2i)\, f(z)$

$$= \lim_{z\to 2i}(z-2i)\frac{\mathbf{1}}{\mathbf{z(z+2i)(z-2i)}}$$

$$= \lim_{z \to 2i} \frac{\mathbf{1}}{\boldsymbol{z(z+2i)}} \quad = \frac{\mathbf{1}}{\mathbf{-8}}$$

$\therefore R_2 = -\frac{\mathbf{1}}{\mathbf{8}}$

If R_3 is a residue of f(z) at simple pole z= 2i then $R_3 = \lim_{z \to -2i} (\mathrm{z} + 2\mathrm{i})\, \mathrm{f(z)}$

$$= \lim_{z \to -2i} (\mathrm{z} + 2\mathrm{i}) \frac{\mathbf{1}}{\boldsymbol{z(z+2i)(z-2i)}}$$

$$= \lim_{z \to -2i} \frac{\mathbf{1}}{\boldsymbol{z(z-2i)}} \quad = \frac{\mathbf{1}}{\mathbf{-8}}$$

$\therefore R_3 = -\frac{\mathbf{1}}{\mathbf{8}}$

$\therefore$ By C. R. Thm c$\int f(z)dz$ = c$\int \frac{\mathbf{1}}{\boldsymbol{z(z^2+4)}}$ dz = 2πi $(R_1+R_2+R_3)$ = $2\pi \mathrm{i}(\frac{\mathbf{1}}{\mathbf{4}} - \frac{\mathbf{1}}{\mathbf{8}} - \frac{\mathbf{1}}{\mathbf{8}})$= 2πi $(\mathbf{0})$ **= 0**

(ii) Now given intergal c$\int f(z)dz$ = c$\int \frac{\boldsymbol{z^2-4}}{\boldsymbol{z(z^2+9)}}$ dz where f(z) $= \frac{\boldsymbol{z^2-4}}{\boldsymbol{z(z^2+9)}} = \frac{\boldsymbol{z^2-4}}{\boldsymbol{z(z+3i)(z-3i)}}$ and C is lzl = 1 Clearly z =0 , $z = 3i\ and\ z = -3i\ are$ simple poles for which only z=0 lie inside the circle lzl = 1 .

$\therefore$ we have to calculate residues at all the pole z=0.

If R_1 is residue of f(z) at pole z= 0 then $R_1 = \lim_{z \to 0} \mathrm{z\, f(z)} = \lim_{z \to 0} \mathrm{z} \frac{\boldsymbol{z^2-4}}{\boldsymbol{z(z^2+9)}}$

$$= \lim_{z \to 0} \frac{\mathbf{-4}}{\boldsymbol{(z^2+9)}} \quad = \frac{\mathbf{1}}{\mathbf{9}}$$

$\therefore R_1 = \frac{\mathbf{-4}}{\mathbf{9}}$

$\therefore$ By C. R. Thm c$\int f(z)dz$ = c$\int \frac{\boldsymbol{z^2-4}}{\boldsymbol{z(z^2+9)}}$ dz = 2πi (R_1) = $2\pi \mathrm{i}(\frac{\mathbf{-4}}{\mathbf{9}})$= $\frac{\mathbf{-8}}{\mathbf{9}}\pi$i

(iv) c$\int \frac{\boldsymbol{1-2z}}{\boldsymbol{z(z-1)(z-2)}}$ **dz where C; lzl =3 (2011, 2015, 2016)**

(v) **(iv)** c$\int \frac{\boldsymbol{3z-1}}{\boldsymbol{(z^3-z)}}$ **dz where C; lzl =2 (2015)**

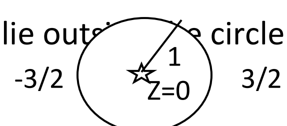
(v) c$\int \frac{\boldsymbol{dz}}{\boldsymbol{z^3(z-1)}}$ **where C; lzl =2 (2012, 2015)**

Try above three example as exercise

(vi) c$\int \frac{\boldsymbol{dz}}{\boldsymbol{(4z^2-9)}}$ **where C;(a) lzl =I (b) lz-1l =1 (2014, 2016)**

Soln: (a) Now given intergal c$\int f(z)dz$ = c$\int \frac{\mathbf{1}}{\mathbf{(4z^2-9)}}$ dz where f(z) $= \frac{\mathbf{1}}{\mathbf{(4z^2-9)}} = \frac{\mathbf{1}}{\mathbf{(2z+3)(2z-3)}}$

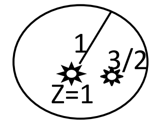
and C is lzl = 1 Clearly $z = \frac{-3}{2}\ and\ z = \frac{3}{2}\ are$ simple poles and both lie outside the circle lzl=1 .



$\therefore$ function is analytic inside C and hence by Cauchy's Theorem c$\int f(z)dz$= 0

(b) Now given intergal $c\int f(z)dz$ = $c\int \frac{\mathbf{1}}{(\mathbf{4z^2-9})}$ dz where f(z) $=\frac{\mathbf{1}}{(\mathbf{4z^2-9})} = \frac{\mathbf{1}}{(\mathbf{2z+3})(\mathbf{2z-3})}$

$=\frac{\mathbf{1}}{\mathbf{4}\left(\boldsymbol{z}+\frac{3}{2}\right)(\boldsymbol{z}-\frac{3}{2})}$



and C is lz-1l = 1

Clearly $z = \frac{-3}{2}$ and $z = \frac{3}{2}$ are simple poles of f(z) for which z $=\frac{3}{2}$ lie inside the circle lz-1l=1

[distance between $\frac{-3}{2}$ and centre (1,0) is $\sqrt{(\frac{-3}{2}-\mathbf{1})^2}$ = $\frac{5}{2}$ >radius 1, ∴ z=$\frac{-3}{2}$ lies outside C]

∴ we have to calculate residues at all the pole z=$\frac{3}{2}$.

If R_1 is residue of f(z) at pole z= $\frac{3}{2}$ then $R_1 = \lim_{z\to\frac{3}{2}}(z-\frac{3}{2})\,f(z)$

$$= \lim_{z\to\frac{3}{2}}(z-\frac{3}{2})\,\frac{\mathbf{1}}{\mathbf{4}\left(\boldsymbol{z}+\frac{3}{2}\right)(\boldsymbol{z}-\frac{3}{2})}$$

$$= \lim_{z\to\frac{3}{2}} \frac{\mathbf{1}}{\mathbf{4}\left(\boldsymbol{z}+\frac{3}{2}\right)} = \frac{\mathbf{1}}{\mathbf{12}}$$

∴ $R_1 = \frac{\mathbf{1}}{\mathbf{12}}$

∴ By C. R. Thm $c\int f(z)dz$ = $c\int\frac{\mathbf{1}}{(\mathbf{4z^2-9})}$ dz = 2πi (R_1) = 2πi($\frac{\mathbf{1}}{\mathbf{12}}$)= $\frac{\mathbf{1}}{\mathbf{6}}\pi$i

HOME WORK

(vii) $c\int\frac{z+4}{z^2+2z+5}$ dz where C; lz+l l =2 (2017) (viii) $c\int\frac{z+4}{z^2(z-1)}$ dz where C; lz l =3 (2018)

(viii) Prove that) $c\int\frac{3z-1}{(z-3)(z+1)}$ dz = $6\pi i$ where C; lz l =4 (2018)

Proof: Now given intergal $c\int f(z)dz$ = $c\int\frac{\mathbf{3z-1}}{(\mathbf{z-3})(\mathbf{z+1})}$ dz where f(z) $=\frac{\mathbf{3z-1}}{(\mathbf{z-3})(\mathbf{z+1})}$ and C is lzl = 4

Clearly $z = 3$ and $z = -1$ are simple poles and both lie intside the circle lzl=4 .

∴ we have to calculate residues at all the pole z=3 and 1 both.

If R_1 is residue of f(z) at pole z= 3 then $R_1 = \lim_{z\to 3}(z-3)\,f(z)$

$$= \lim_{z\to 3}(z-3)\,\frac{\mathbf{3z-1}}{(\mathbf{z-3})(\mathbf{z+1})}$$

$$= \lim_{z\to 3}\frac{\mathbf{3z-1}}{(\mathbf{z+1})} = \frac{\mathbf{8}}{\mathbf{4}} = 2$$

∴ $R_1 = \mathbf{2}$

If R_2 is residue of f(z) at pole z= -1 then $R_1 = \lim_{z\to -1}(z+1)\,f(z)$

$$= \lim_{z\to -1}(z+1)\,\frac{\mathbf{3z-1}}{(\mathbf{z-3})(\mathbf{z+1})}$$

$$= \lim_{z\to -1}\frac{\mathbf{3z-1}}{(\mathbf{z-3})} = \frac{\mathbf{-4}}{\mathbf{-4}} = 1$$

∴ R_2 =1

∴ By C. R. Thm $c\int f(z)dz$ = $c\int\frac{\mathbf{3z-1}}{(\mathbf{z-3})(\mathbf{z+1})}$ dz = 2πi (R_1+R_2) = 2πi($\mathbf{2}+\mathbf{1}$)= $\mathbf{6}\pi$i

15. Find residues of f(z) =$\frac{1}{z(z^2 3z+2)}$ at z= 0, 1 and -2 and hence evaluate c$\oint$ f(z) dz where c :lzl = 3.

Soln:

II. Evaluation of real integral of the type $\int_0^{2\pi} f(sin\theta, cos\theta)d\theta$

Contour Integration: Evaluation of integral of above type by our usual real integral is sometimes tedious, hence in such cases we reduce above integral to c$\oint f(z)dz$ taken around the closed contour C, and thus is called contour integration.

[In PUC , we already come across the examples of the type $\int_0^{2\pi} \frac{1}{a+bcos\theta}d\theta$, $\int_0^{2\pi} \frac{1}{a+bsin\theta}d\theta$ **etc.** Such type of examples can be solved easily using contour integration.]

Procedure for evaluation above integral:

Consider given integral $\int_0^{2\pi} f(sin\theta, cos\theta)d\theta$-------------------(1)

Take substitution $e^{i\theta}$ = z so that $e^{-i\theta}$ = $\frac{1}{z}$ and cosθ=$\frac{1}{2}$ (z + $\frac{1}{z}$) and sinθ= $\frac{1}{2i}$ (z - $\frac{1}{z}$)

and also $e^{i\theta}$ *i* dθ = dz => dθ= $\frac{dz}{i\,e^{i\theta}}$ = $\frac{dz}{i\,z}$

∴ dθ= $\frac{dz}{i\,z}$ and θ =0 to 2π is for circle C : lzl =1

By all these substitution (1) becomes

$\int_0^{2\pi} f(sin\theta, cos\theta)d\theta$= c$\oint f\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right) dz$

= c$\oint f(z)dz$, anyhow terms inside the brocket are functions of z.

And the integral c$\oint f(z)dz$ can be evaluated by C.R. theorem as in previous examples taken around unit **circle C: lzl =1**

NOTE: For the examples of above type this substitution is fixed and is C is also always unit circle lzl=1.

Examples:(compulsory one example for 5 marks)

(i) If N^r is constant and D^r either in terms of sinθ or cosθ

1. **Using contour integration , evaluate $_0\int^{2\pi} \frac{d\theta}{5+4\cos\theta}$ (2014, 2015, 2018)**

Soln: Given integral ${}_0\int^{2\pi} \frac{d\theta}{5+4\cos\theta)}$ -----------------(1)

Put $e^{i\theta} = z$ so that $d\theta = \frac{dz}{i\,z}$ (for all examples it is same, so we remember this)

And $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, C is |z| =1

Then integral (1) becomes ${}_0\int^{2\pi} \frac{d\theta}{5+4\cos\theta} = {}_c\int \frac{\frac{dz}{iz}}{(5+4\frac{1}{2}\left(z+\frac{1}{z}\right))} = {}_c\int \frac{\frac{dz}{iz}}{5+2\left(\frac{z^2+1}{z}\right)}$

$$= {}_c\int \frac{\frac{dz}{iz}}{\left(\frac{5z+2z^2+2}{z}\right)} = {}_c\int \frac{dz}{iz\left(\frac{5z+2z^2+2}{z}\right)}$$

$$= \frac{1}{i}\,{}_c\int \frac{dz}{2z^2+5z+2} = \frac{1}{i}\,{}_c\int f(z)dz \quad \text{---------(2)}$$

Where $f(z) = \frac{1}{2z^2+5z+2} = \frac{1}{(2z+1)(z+2)} = \frac{1}{2(z+\frac{1}{2})(z+2)}$ (D^r is having linear factors)

Clearly $z = -\frac{1}{2}$ and $z = -2$ are simple poles of f(z) for which $z = -\frac{1}{2}$ lies inside the circle |z|=1

∴ calculate residue at $z = -\frac{1}{2}$

If R_1 is the residue of f(z) at pole $z = -\frac{1}{2}$ **(simple pole)**

Then $R_1 = \lim_{z\to-\frac{1}{2}} \left(z+\frac{1}{2}\right) f(z) = \lim_{z\to-\frac{1}{2}} \left(z+\frac{1}{2}\right)\frac{1}{2(z+\frac{1}{2})(z+2)} = \lim_{z\to-\frac{1}{2}} \frac{1}{2(z+2)}$

$$= \frac{1}{2(-\frac{1}{2}+2)} = \frac{1}{2(\frac{3}{2})} = \frac{\mathbf{1}}{\mathbf{3}}$$

∴ $R_1 = \frac{\mathbf{1}}{\mathbf{3}}$

By C.R. Thm we have ${}_c\int f(z)dz = 2\pi i\, R_1 = 2\pi i(\frac{\mathbf{1}}{\mathbf{3}})$**---------------(3)**

Substitute (3) in (2) then given integral ${}_0\int^{2\pi} \frac{d\theta}{5+4\cos\theta} = \frac{1}{i}\,{}_c\int f(z)dz = \frac{1}{i}\left(\frac{2\pi i}{3}\right) = \frac{2\pi}{3}$

Hence ${}_0\int^{2\pi} \frac{d\theta}{5+4\cos\theta} = \frac{2\pi}{3}$ (answer should be in terms of real no. as integral is real)

(Same procedure for examples of these types)

2. **Using contour integration prove that ${}_0\int^{2\pi} d\theta/(a+b\cos\theta) = 2\pi/\sqrt{a^2-b^2}$ where |b|<a. (2012)**

Soln: Given integral ${}_0\int^{2\pi} \frac{d\theta}{a+b\cos\theta}$ -----------------(1)

Put $e^{i\theta} = z$ so that $d\theta = \frac{dz}{i\,z}$ (for all examples it is same, so we remember this)

and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, C is |z| =1

Then integral (1) becomes ${}_0\int^{2\pi} \frac{d\theta}{a+b\cos\theta} = {}_c\int \frac{\frac{dz}{iz}}{a+b\frac{1}{2}\left(z+\frac{1}{z}\right))} = {}_c\int \frac{\frac{dz}{iz}}{a+b\left(\frac{z^2+1}{2z}\right)}$

$$= {}_c\int \frac{\frac{dz}{iz}}{\left(\frac{2az+bz^2+b}{2z}\right)} = {}_c\int \frac{dz}{iz\left(\frac{2az+bz^2+b}{2z}\right)}$$

$$= \frac{2}{i}\,{}_c\int \frac{dz}{bz^2+2az+b} = \frac{2}{i}\,{}_c\int f(z)dz \quad \text{---------(2)}$$

Where f(z) = $\frac{1}{bz^2+2az+b} = \frac{1}{b\,(z^2+2\frac{a}{b}z+1)} = \frac{1}{b(z-\alpha)(z-\beta)}$ (D^r is general eqn. so let the factors be in general) where $\alpha = \frac{-2\frac{a}{b}+\sqrt{(2\frac{a}{b})^2-4}}{2} = \frac{-2\frac{a}{b}+\sqrt{(2\frac{a}{b})^2-4}}{2} = \frac{-2a+\sqrt{4a^2-4b^2}}{2b} = \frac{-a+\sqrt{a^2-b^2}}{b}$

& $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$ (b'cz irrational roots occur in conjugate pairs)

[these roots are obtained by formula x = $\frac{-b+\sqrt{b^2-4ac}}{2a}$ method, for factors of this type we use the same procedure]

Clearly z = α and z = β are simple poles of f(z) for which z = α lies inside the circle |z|=1

[b'cz in the example it is given that **|b|<a, ∴** $\left|\frac{-a+\sqrt{a^2-b^2}}{b}\right|$ **<1 , i.e** $|\alpha| < 1$ *i.e* **distance between** α and centre is < 1, *but* $|\beta| > 1$ **as** $\left|\frac{-a-\sqrt{a^2-b^2}}{b}\right| > 1$ **]**

∴ calculate residue at z = α

If R_1 is the residue of f(z) at pole z = α **(simple pole)**

Then $R_1 = \lim_{z\to\alpha}(z-\alpha)\,f(z) = \lim_{z\to\alpha}(z-\alpha)\frac{1}{b(z-\alpha)(z-\beta)} = \lim_{z\to\alpha}\frac{1}{b(z-\beta)}$

$= \frac{1}{b\,(\alpha-\beta)} = \frac{1}{b\,(\frac{2\sqrt{a^2-b^2}}{b})} = \frac{\mathbf{1}}{\mathbf{2\sqrt{a^2-b^2}}}$

∴ $R_1 = \frac{\mathbf{1}}{\mathbf{2\sqrt{a^2-b^2}}}$

By C.R. Thm we have $_C\int f(z)dz = 2\pi i\, R_1 = 2\pi i\frac{\mathbf{1}}{\mathbf{2\sqrt{a^2-b^2}}} = \frac{\pi i}{\sqrt{a^2-b^2}}$ **--------(3)**

Substitute (3) in (2) then given integral $_0\int^{2\pi}\frac{d\theta}{a+b\cos\theta} = \frac{2}{i}\,_C\int f(z)dz = \frac{2}{i}\left(\frac{\pi i}{\sqrt{a^2-b^2}}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}$

Hence $_0\int^{2\pi}\frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ (answer should be in terms of real no. as integral is real)

[In first example if we put a=5, b= 4, we get answer $_0\int^{2\pi}\frac{d\theta}{5+4\cos\theta} = \frac{2\pi}{\sqrt{5^2-4^2}} = \frac{2\pi}{3}$ **which is true]**

3. Using contour integration prove that $_0\int^{\pi} d\theta/(a+\cos\theta) = \pi/\sqrt{a^2-1}$ where a>1.

Soln: Given integral $_0\int^{2\pi}\frac{d\theta}{a+\cos\theta}$ -----------------(1)

Put $e^{i\theta} = z$ so that $d\theta = \frac{dz}{i\,z}$ (for all examples it is same, so we remember this)

and $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$, C is |z| =1

Then integral (1) becomes $_0\int^{2\pi}\frac{d\theta}{a+\cos\theta} = \,_C\int\frac{\frac{dz}{iz}}{a+\frac{1}{2}\left(z+\frac{1}{z}\right)} = \,_C\int\frac{\frac{dz}{iz}}{a+\left(\frac{z^2+1}{2z}\right)}$

$= \,_C\int\frac{\frac{dz}{iz}}{\left(\frac{2az+z^2+1}{2z}\right)} = \,_C\int\frac{dz}{iz\left(\frac{az+z^2+1}{2z}\right)}$

$= \frac{2}{i}\, {}_{C}\!\int \frac{dz}{z^2+2az+1} = \frac{2}{i}\, {}_{C}\!\int f(z)dz$ ---------(2)

Where f(z) = $\frac{1}{z^2+2az+1} = \frac{1}{(z^2+2\,az+1)} = \frac{1}{(z-\alpha)(z-\beta)}$ (D^r is general eqn. so let the factors be in general) where $\alpha = \frac{-2a+\sqrt{(2a)^2-4}}{2} = \frac{-2a+\sqrt{(2a)^2-4}}{2} = \frac{-2a+\sqrt{4a^2-4}}{2} = -a+\sqrt{a^2-1}$

& $\beta = -a-\sqrt{a^2-1}$ (b'cz irrational roots occur in conjugate pairs)

[these roots are obtained by formula x = $\frac{-b+\sqrt{b^2-4ac}}{2a}$ method, for factors of this type we use the same procedure]

Clearly z = α and z = β are simple poles of f(z) for which z = α lies inside the circle |z|=1

[b'cz in the example it is given that **1<a,** $\therefore \left|\frac{-a+\sqrt{a^2-1}}{1}\right|$ **<1 , i.e** $|\alpha| < 1$ *i.e* **distance between** α and centre is < 1, *but* $|\beta| > 1$ **as** $\left|\frac{-a-\sqrt{a^2-1}}{1}\right| > 1$]

∴ calculate residue at z = α

If R_1 is the residue of f(z) at pole z = α **(simple pole)**

Then $R_1 = \lim_{z\to\alpha}(z-\alpha)\, f(z) = \lim_{z\to\alpha}(z-\alpha)\frac{1}{(z-\alpha)(z-\beta)} = \lim_{z\to\alpha}\frac{1}{(z-\beta)}$

$= \frac{1}{(\alpha-\beta)} = \frac{\mathbf{1}}{\mathbf{2\sqrt{a^2-1}}}$

$\therefore R_1 = \frac{\mathbf{1}}{\mathbf{2\sqrt{a^2-1}}}$

By C.R. Thm we have ${}_{C}\!\int f(z)dz = 2\pi i\, R_1 = 2\pi i\frac{\mathbf{1}}{\mathbf{2\sqrt{a^2-1}}} = \frac{\pi i}{\sqrt{a^2-1}}$ **--------(3)**

Substitute (3) in (2) then given integral ${}_0\!\int^{2\pi} \frac{d\theta}{a+\cos\theta} = \frac{2}{i}\, {}_{C}\!\int f(z)dz = \frac{2}{i}\left(\frac{\pi i}{\sqrt{a^2-1}}\right)$

$= \frac{2\pi}{\sqrt{a^2-b^2}}$

Hence ${}_0\!\int^{2\pi} \frac{d\theta}{a+\cos\theta} = \frac{2\pi}{\sqrt{a^2-1}}$

[Same as example (2), in the place of b we have to put b=1]

4. **Using contour integration prove that ${}_0\!\int^{2\pi} d\theta/(1+a\cos\theta) = 2\pi/\sqrt{1-a^2}$ where |a|<1.**
5. **Using contour integration prove that ${}_0\!\int^{2\pi} d\theta/(2+\cos\theta) = 2\pi/\sqrt{3}$ (2013, 2015)**
   HOME work (same as above examples, only values a and b are different)
6. **Evaluate ${}_0\!\int^{2\pi} d\theta/(a+b\sin\theta)$ by contour integration where |a| <1**

Soln: Given integral ${}_0\!\int^{2\pi} \frac{d\theta}{a+b\sin\theta}$ -----------------(1)

Put $e^{i\theta} = z$ so that $d\theta = \frac{dz}{iz}$ (for all examples it is same, so we remember this)

and $\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$, C is |z| =1

Then integral (1) becomes ${}_0\!\int^{2\pi} \frac{d\theta}{a+b\sin\theta} = {}_{C}\!\int \frac{\frac{dz}{iz}}{a+b\frac{1}{2i}\left(z-\frac{1}{z}\right)} = {}_{C}\!\int \frac{\frac{dz}{iz}}{a+b\left(\frac{z^2-1}{2iz}\right)}$

$$= c\oint \frac{\frac{dz}{iz}}{\left(\frac{2aiz+bz^2-b}{2iz}\right)} = c\oint \frac{dz}{iz\left(\frac{2aiz+bz^2-b}{i2z}\right)}$$

$$= 2\, c\oint \frac{dz}{bz^2+2aiz-b} = 2\, c\oint f(z)dz \quad \text{---------(2)}$$

Where $f(z) = \frac{1}{bz^2+2aiz-b} = \frac{1}{b\,(z^2+2\frac{ai}{b}z-1)} = \frac{1}{b(z-\alpha)(z-\beta)}$ (D^r is general eqn. so let the factors be in general) where $\alpha = \frac{-2\frac{ai}{b}+\sqrt{(2\frac{ai}{b})^2+4}}{2} = \frac{-2\frac{a}{b}+\sqrt{(2\frac{ai}{b})^2+4}}{2} = \frac{-2ia+\sqrt{4(ai)^2+4b^2}}{2b} = \frac{-2ia+2i\sqrt{a^2-b^2}}{2b}$

$= i\,\frac{-a+\sqrt{a^2-b^2}}{b}$ (b'cz irrational roots occur in conjugate pairs)

& $\beta = i\,\frac{-a-\sqrt{a^2-b^2}}{b}$

[these roots are obtained by formula $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ method, for factors of this type we use the same procedure]

Clearly z = α and z = β are simple poles of f(z) for which z = α lies inside the circle lzl=1

[b'cz in the example it is given that **lbl<a, ∴** $\left|\frac{-a+\sqrt{a^2-b^2}}{b}\right|$ **<1 , i.e** $|\alpha| < 1$ *i.e* **distance between** α and centre is < 1, *but* $|\beta| > 1$ **as** $\left|\frac{-a-\sqrt{a^2-b^2}}{b}\right| > 1$ **and lil=1]**

∴ calculate residue at z = α

If R_1 is the residue of f(z) at pole z = α **(simple pole)**

Then $R_1 = \lim_{z\to\alpha}(z-\alpha)\, f(z) = \lim_{z\to\alpha}(z-\alpha)\frac{1}{b(z-\alpha)(z-\beta)} = \lim_{z\to\alpha}\frac{1}{b(z-\beta)}$

$= \frac{1}{b\,(\alpha-\beta)} = \frac{1}{b\,(\frac{2i\sqrt{a^2-b^2}}{b})} = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{i}\sqrt{a^2-b^2}}$

∴ $R_1 = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{i}\sqrt{a^2-b^2}}$

By C.R. Thm we have $c\oint f(z)dz = 2\pi i\, R_1 = 2\pi i\frac{1}{2i\sqrt{a^2-b^2}} = \frac{\pi}{\sqrt{a^2-b^2}}$ **--------(3)**

Substitute (3) in (2) then given integral $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = 2\, c\oint f(z)dz = 2\left(\frac{\pi}{\sqrt{a^2-b^2}}\right) =$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

Hence $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ (answer should be in terms of real no. as integral is real)

7. Evaluate $\int_0^{2\pi} \frac{d\theta}{\frac{5}{4}+\sin\theta}$ **or** $\int^{2\pi} \frac{4d\theta}{5+4\sin\theta}$ **(2009)**

Soln: In above example put a = $\frac{5}{4}$, **b = 1, we get** $\int_0^{2\pi} \frac{d\theta}{\frac{5}{4}+\sin\theta} = \frac{2\pi}{\sqrt{(\frac{5}{4})^2-1^2}} = \frac{2\pi}{\sqrt{\frac{9}{16}}} = \frac{8\pi}{3}$

Try it as home work

8. Using contour integration prove that $\int_0^{2\pi}$ [cos2θ / (5+4cosθ)] dθ = π/6.

Soln: In this example N^r is not constant it is a function of cosθ, to solve example of this type starting procedure is different

We have $e^{2i\theta} = \cos 2\theta + i\sin 2\theta$ $\quad\therefore \cos 2\theta$ = Real part of $e^{2i\theta}$

$$\therefore \frac{\cos 2\theta}{5+4\sin\theta} = \text{R.P of } \frac{e^{2i\theta}}{5+4\sin\theta}$$

$\therefore$ Given integral ${}_0\int^{2\pi} \frac{\cos 2\theta d\theta}{5+4\cos\theta)}$ = R. P of ${}_0\int^{2\pi} \frac{e^{2i\theta}d\theta}{5+4\cos\theta)}$ = R. P of ${}_0\int^{2\pi} \frac{(e^{i\theta})^2 d\theta}{5+4\cos\theta)}$-----------------(1)

Put $e^{i\theta}$ = z so that $d\theta = \frac{dz}{i\,z}$ (for all examples it is same, so we remember this)

And $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, C is lzl =1

Then integral (1) becomes${}_0\int^{2\pi} \frac{\cos 2\theta d\theta}{5+4\cos\theta)} =$ R. P of ${}_0\int^{2\pi} \frac{(e^{i\theta})^2 d\theta}{5+4\cos\theta}$ = R. P of ${}_c\int \frac{z^2 \frac{dz}{iz}}{(5+4\frac{1}{2}\left(z+\frac{1}{z}\right))}$

= R. P of ${}_c\int \frac{z^2 \frac{dz}{iz}}{5+2\left(\frac{z^2+1}{z}\right)}$ (same as example ${}_0\int^{2\pi} \frac{d\theta}{5+4\cos\theta)}$ but only change is in N^r , extra term z^2)

= R. P of ${}_c\int \frac{z^2 \frac{dz}{iz}}{\left(\frac{5z+2z^2+2}{z}\right)}$ = R. P of ${}_c\int \frac{z^2\ dz}{iz\left(\frac{5z+2z^2+2}{z}\right)}$ = R. P of $\frac{1}{i}\,{}_c\int \frac{z^2\ dz}{2z^2+5z+2} = \frac{1}{i}\,{}_c\int f(z)dz$ ---------(2)

Where f(z) = $\frac{z^2}{2z^2+5z+2} = \frac{z^2}{(2z+1)(z+2)} = \frac{z^2}{2(z+\frac{1}{2})(z+2)}$ (D^r is having linear factors)

Clearly z = $-\frac{1}{2}$ and z = - 2 are simple poles of f(z) for which z = $-\frac{1}{2}$ lies inside the circle lzl=1

$\therefore$ calculate residue at z = $-\frac{1}{2}$

If R_1 is the residue of f(z) at pole z = $-\frac{1}{2}$ **(simple pole)**

Then $R_1 = \lim_{z\to-\frac{1}{2}} (z+\frac{1}{2})\, f(z) = \lim_{z\to-\frac{1}{2}} \left(z+\frac{1}{2}\right)\frac{z^2}{2(z+\frac{1}{2})(z+2)} = \lim_{z\to-\frac{1}{2}} \frac{z^2}{2(z+2)}$

$$= \frac{\frac{1}{4}}{2(-\frac{1}{2}+2)} = \frac{1}{8(\frac{3}{2})} = \frac{\mathbf{1}}{\mathbf{12}}$$

$\therefore R_1 = \frac{\mathbf{1}}{\mathbf{12}}$

By C.R.of hm we have ${}_c\int f(z)dz$ =2πI R_1 = $2\pi i(\frac{\mathbf{1}}{\mathbf{12}}) = \frac{\pi i}{\mathbf{6}}$**---------------(3)**

Substitute (3) in (2) then given integral ${}_0\int^{2\pi} \frac{\cos 2\theta\ d\theta}{5+4\cos\theta}$ =R. P.of $\frac{1}{i}\,{}_c\int f(z)dz$ = R. P.of $\frac{1}{i}(\frac{\pi i}{6})$

= R. P.of $\frac{\pi}{6} = \frac{\pi}{6}$ (b'cz real part of real no. is itself)

Hence ${}_0\int^{2\pi} \frac{\cos 2\theta\ d\theta}{5+4\cos\theta} = \frac{\pi}{6}$ (answer should be in terms of real no. as integral is real)

9. **Prove that ${}_0\int^{\pi} [\frac{1+2\cos\theta}{5+4\sin\theta}]\, d\theta = 0$**

Soln: In this example N^r is not constant it is a function of cosθ, to solve example of this type starting procedure is different

We have $e^{i\theta} = \cos\theta + i\sin\theta$ $\therefore$ $1+2e^{i\theta} = (1+2\cos\theta) + i\sin\theta$ $\therefore$ $1+2\cos\theta$ = R. P of $(1+2e^{i\theta})$

$\therefore$ $\frac{1+2\cos\theta}{5+4\sin\theta}$ **= R.P of** $\frac{1+2e^{i\theta}}{5+4\sin\theta}$

$\therefore$ Given integral ${}_0\int^{2\pi} \frac{1+2\cos\theta d\theta}{5+4\cos\theta)}$ = R. P of ${}_0\int^{2\pi} \frac{1+2e^{i\theta}d\theta}{5+4\cos\theta)}$ -----------------(1)

Put $e^{i\theta} = z$ so that $d\theta = \frac{dz}{iz}$ (for all examples it is same, so we remember this)

And $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, C is |z| =1

Then integral (1) becomes ${}_0\int^{2\pi} \frac{1+2\cos\theta d\theta}{5+4\cos\theta)} =$ R. P of ${}_0\int^{2\pi} \frac{1+2e^{i\theta}d\theta}{5+4\cos\theta}$ = R. P of ${}_c\int \frac{(1+2z)\frac{dz}{iz}}{(5+4\frac{1}{2}\left(z+\frac{1}{z}\right))}$

= R. P of ${}_c\int \frac{(1+2z)\frac{dz}{iz}}{5+2\left(\frac{z^2+1}{z}\right)}$

= R. P of ${}_c\int \frac{(1+2z)\frac{dz}{iz}}{\left(\frac{5z+2z^2+2}{z}\right)}$ = R. P of ${}_c\int \frac{(1+2z)dz}{iz\left(\frac{5z+2z^2+2}{z}\right)}$ = R. P of $\frac{1}{i}$ ${}_c\int \frac{(1+2z)dz}{2z^2+5z+2}$

= R. P of $\frac{1}{i}$ ${}_c\int f(z)dz$ ---------(2)

Where $f(z) = \frac{(1+2z)}{2z^2+5z+2} = \frac{\cancel{(1+2z)}}{\cancel{(2z+1)}(z+2)} = \frac{1}{(z+2)}$

Clearly z = - 2 are simple poles of f(z) for which lies outside the circle |z|=1

$\therefore$ **By Cauchy's Thm** ${}_c\int f(z)dz = 0$

From (2) given integral is ${}_0\int^{2\pi} \frac{1+2\cos\theta d\theta}{5+4\cos\theta}$ =R. P.of $\frac{1}{i}$ ${}_c\int f(z)dz$ = R. P.of $\frac{1}{i}(0)$ =0

Hence ${}_0\int^{2\pi} \frac{1+2\cos\theta\, d\theta}{5+4\cos\theta}$ **= 0**

10.Using contour integration evaluate $0\int^{2\pi}$ **[cos3θ / (5+4cosθ)] dθ (2017)**
HOME work

10. Prove that ${}_0\int^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\, d\theta = 2\pi/n!$ (2015)

Soln: This example is again little different

Let $\alpha = (\sin\theta - n\theta)$ then $e^{\cos\theta}\cos(\sin\theta-n\theta) = e^{\cos\theta}\cos\alpha$ = R. P of $e^{\cos\theta}(\cos\alpha + i\sin\alpha)$

= R. P of $(e^{\cos\theta}e^{i\alpha})$ = R. P of $(e^{\cos\theta}e^{i(\sin\theta-n\theta)})$ = R. P of $(e^{\cos\theta}e^{i\sin\theta - in\theta})$

= R. P of $(e^{\cos\theta + i\sin\theta}e^{-in\theta})$ = R. P of $e^{(e^{i\theta})}\ e^{-in\theta}$ = R. P of $e^{(e^{i\theta})}\ (e^{i\theta})^{-n}$

Given Integral is ${}_0\int^{2\pi} e^{\cos\theta}\cos(\sin\theta-n\theta)\, d\theta$ = R. P of ${}_0\int^{2\pi} e^{(e^{i\theta})}\ (e^{i\theta})^{-n}\ d\theta$ ---------(1)

Put $e^{i\theta} = z$ so that $d\theta = \frac{dz}{iz}$ then integral (1) becomes

${}_0\int^{2\pi} e^{\cos\theta}\cos(\sin\theta-n\theta)\, d\theta$ = R. P of ${}_0\int^{2\pi} e^{(e^{i\theta})}\ (e^{i\theta})^{-n}\ d\theta$ = R.P. of ${}_c\int e^{z} z^{-n}\frac{dz}{iz}$ where

C is unit circle |z|=1

= R.P. of $\frac{1}{i}\,{}_c\oint\; e^z\; \frac{dz}{z^{n+1}}$ = R.P. of $\frac{1}{i}\,{}_c\oint\; \frac{e^z}{z^{n+1}}\,dz$ (we have done example of this type)

= R.P. of $\frac{1}{i}\,{}_c\oint\; f(z)dz$ --------------(2)

Where f(z) $=\frac{e^z}{z^{n+1}}$ Clearly z =0 is a pole of order (n+1) which is inside the circle |z| =1 .

If R_1 is residue of f(z) at z= 0 then $R_1 = \frac{1}{(n+1-1)!}\lim_{z\to 0}\frac{d^n}{dz^n}(z-0)^n\, f(z)$

$=\frac{1}{n!}\lim_{z\to 0}\frac{d^n}{dz^n}\cancel{(z)^n}\frac{e^z}{\cancel{(z)^n}}$

$=\frac{1}{n!}\lim_{z\to\pi}\frac{d^n}{dz^n}\,e^z = \frac{1}{n!}\lim_{z\to 0}\, e^z = \frac{1}{n!}$

= 1

∴ By C. R. Thm ${}_c\oint f(z)dz = {}_c\oint \frac{e^z}{z^{n+1}}$ **dz** = 2πi (R_1) = $2\pi i(\frac{1}{n!}) = \frac{2\pi i}{n!}$-----------------(3)

Then from (2) and (3) given integral becomes

$_0\int^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\, d\theta$ = R. P of $\frac{1}{i}\,{}_c\oint\; f(z)dz$ = R. P of $\frac{1}{i}(\frac{2\pi i}{n!})$ = R. P of $\frac{2\pi}{n!}$

$= \frac{2\pi}{n!}$ (b'cz real part of real is itself)

∴ $\boldsymbol{_0\int^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\, d\theta = \frac{2\pi}{n!}}$

This complets 2nd type of examples

III. Evaluation of real integral of the type $\int_{-\infty}^{\infty} f(x)dx$ provided poles are not real, i.e they are only in terms imaginary)

[for example $\int_{-\infty}^{\infty} \frac{e^x}{(x+2)^2} dx$ cannot be solved as, pole is x= - 2 is real but example $\int_{-\infty}^{\infty} \frac{x}{x^2+4} dx$ can be solved as poles are x=$\pm$2i, which are imaginary

Similarly $\int_{-\infty}^{\infty} \frac{cosx}{x^2+2x+3} dx$ can be solved as poles are x=$-2 \pm i\,2\sqrt{2}$, which are imaginary]

To solve examples of these types we need one lemma, called Jordan's Lemma

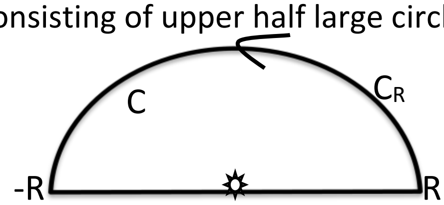
Statement for Jordan's Lemma(Important for 2 marks): **If f(z) →0 uniformly as |z| → ∞ (i.e region tends to hole plane) then $\lim_{R\to\infty} \int_{C_R} e^{imz} f(z)dz$ = 0 where C_R is denotes Semi circle |z| = R, I(z)|>0**

Procedure to solve example of this type :

Given integral $\int_{-\infty}^{\infty} f(x)dx$ steps are as follows

(i) Consider the integral as $\oint f(z)dz$, just replace x by z and write the integral where C is closed contour consisting of upper half large circle C_R : |z|=R and real line from –R to R

(C_R does not look like semi circle, assume it as semi circle)



Closed curve C contains two parts ,C_R Upper half of circle |z|=R and real line from –R to R.
∴ C =C_R + line from –R to R

(ii) Next for f(z) , find poles and calculate residues at poles which lie inside C, let them be R_1, R_2---------

(iii) By C.R. Theorem we have $\oint f(z)dz$ = 2πI (sum of residues) =let it be some value K

i.e $\oint f(z)dz$ = K

i.e $\int_{C_R} f(z)dz$ + $\int_{-R}^{R} f(x)dx$ = K (b'cz C consisting two parts C_R and real line from –R to R)

(iv) **Taking limit as $R \to \infty$ on both sides, we get**

$$\lim_{R\to\infty} {}_{c_R}\!\int f(z)dz + \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \lim_{R\to\infty} K$$

$$\lim_{R\to\infty} {}_{c_R}\!\int f(z)dz + \int_{-\infty}^{\infty} f(x)dx = K \quad \text{(b'cz limit of constant is constant)}$$

$\therefore \quad \int_{-\infty}^{\infty} f(x)dx = K - \lim_{R\to\infty} {}_{c_R}\!\int f(z)dz$ **and integral** ${}_{c_R}\!\int f(z)dz$ **in RHS can be evaluated by Jordans Lemma or by any other method so that in all the examples** $\lim_{R\to\infty} {}_{c_R}\!\int f(z)dz = 0$

$$\therefore \quad \int_{-\infty}^{\infty} f(x)dx = K - 0$$

i.e $\int_{-\infty}^{\infty} f(x)dx = K$ **and** $\int_{0}^{\infty} f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx = \frac{K}{2}$

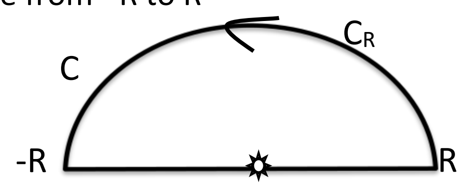
NOTE: This procedure is same for all examples of above type.

Examples: .

1 . Prove that ${}_0\!\int^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$ **by contour integration (2015)**

Soln: Given integral ${}_0\!\int^{\infty} \frac{dx}{1+x^2}$

Consider the integral ${}_c\!\int f(z)dz = {}_c\!\int \frac{dz}{1+z^2}$, taken around the closed contour C consisting of upper half large circle C_R : |z|=R and real line from –R to R



Here f(z) = $\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$

Clearly Z=i, -i are simple poles of f(z) for which Z = i lies inside C (where as z = -i = (0, -1) lies lower part of z-plane but our region is only upper half of z-plane)

$\therefore$ calculate residue at z=i

If R_1 is the residue of f(z) at z =i then R_1 = = $\lim_{z\to i}(z-i)\, f(z)$

$$= \lim_{z\to i}(z-i)\frac{1}{(z+i)(z-i)}$$

$$= \lim_{z\to i} \frac{1}{(z+i)} = \frac{1}{2i}$$

By C.R. Theorem we have ${}_c\!\int f(z)dz = 2\pi i\,(R_1)$

i.e ${}_c\!\int f(z)dz = 2\pi i\,\frac{1}{2i}$

i.e ${}_{c_R}\!\int f(z)dz + \int_{-R}^{R} f(x)dx = \pi$

Taking limit as $R \to \infty$ on both sides, we get

$$\lim_{R\to\infty} {}_{c_R}\!\int f(z)dz + \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \lim_{R\to\infty} \pi$$

$$\lim_{R\to\infty} {}_{c_R}\!\int f(z)dz + \int_{-\infty}^{\infty} f(x)dx = \pi \quad \text{(b'cz limit of constant is constant)}$$

$\therefore \quad \int_{-\infty}^{\infty} f(x)dx = \pi - \lim_{R\to\infty} {}_{C_R}\int f(z)dz$

i.e $\int_{-\infty}^{\infty} \frac{1}{1+x^2} \; dx = \pi - \lim_{R\to\infty} {}_{C_R}\int \frac{1}{1+z^2} \; dz$ ----------------------(1)

Consider $\left| {}_{C_R}\int \frac{1}{1+z^2} \; dz \right| \le {}_{C_R}\int \left|\frac{1}{1+z^2}\right| \; |dz| = {}_{C_R}\int \frac{1}{|z^2+1|} \; |dz|$

$\le \; {}_{C_R}\int \frac{1}{|z|^2-1} \; |dz|$ [b'cz $\frac{1}{|a+b|} \le \frac{1}{|a|-|b|}$]

$= \; {}_{C_R}\int \frac{1}{R^2-1} \; |dz|$ [b'cz [z] = R]

$= \frac{1}{R^2-1} \, {}_{C_R}\int \; |dz|$

$= \; \frac{1}{R^2-1}$ length of the semi circle C_R

$= \frac{1}{R^2-1} (\pi R) \to 0$ as $R \to \infty$

Thus $\lim_{R\to\infty} \left| {}_{C_R}\int \frac{1}{1+z^2} dz \right| = 0$

=> $\lim_{R\to\infty} {}_{C_R}\int \frac{1}{1+z^2} dz = 0$

From (1) given integral becomes $\int_{-\infty}^{\infty} \frac{1}{1+x^2} \; dx = \pi - 0 = \pi$

i. e $2\int_{0}^{\infty} \frac{1}{1+x^2} \; dx = \pi$ [b'cz $\int_{-a}^{a} f(x) \; dx = 2\int_0^a \frac{1}{1+x^2} \; dx \; as \; f(x) is \; even \; fuction$]

=> $\int_{0}^{\infty} \frac{1}{1+x^2} \; dx = \frac{\pi}{2}$

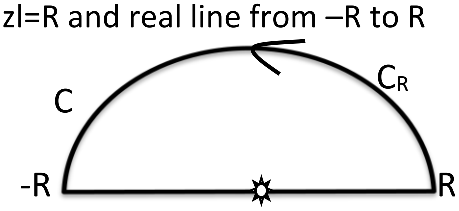
$\therefore \int_{0}^{\infty} \frac{1}{1+x^2} \; dx = \frac{\pi}{2}$

Note: Above example can be solved even by using PUC integration, if the power of D^r increases we cannot evaluate by our PUC integration, so in such cases contour integration is applicable.

2. Prove that $_{-\infty}\int^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$ by contour integration. (2016, 2017) [in this example power of D^r is 2]

Proof: Given integral $_0\int^{\infty} \frac{dx}{(1+x^2)^2}$

Consider the integral $_C\int f(z)dz = {}_C\int \frac{dz}{(1+z^2)^2}$, taken around the closed contour C consisting of upper half large circle C_R : lzl=R and real line from –R to R



Here f(z) = $\frac{1}{(1+z^2)^2} = \frac{1}{((z+i)(z-i))^2} = \frac{1}{(z+i)^2 \; (z-i)^2}$

Clearly Z=i, -i are poles of f(z) of order 2 for which Z = i lies inside C (where as z = -i = (0, -1) lies lower part of z-plane but our region is only upper half of z-plane)

$\therefore$calculate residue at pole z=i (order is 2)

If R_1 is the residue of f(z) at z = i then $R_1 = \frac{1}{(2-1)!} \lim_{z\to i} \frac{d}{dz}(z-i)^2 \; f(z)$

$$= \lim_{z\to i} \frac{d}{dz}(z-i)^2 \frac{1}{(z+i)^2\,(z-i)^2}$$

$$= \lim_{z\to i} \frac{d}{dz}\frac{1}{(z+i)^2} = \lim_{z\to i} \frac{-2}{(z+i)^3} = \frac{-2}{(2i)^3} = \frac{-2}{-8i} = \frac{1}{4i}$$

By C.R. Theorem we have ${}_c\int f(z)dz$ **= 2πi (R₁)**

i.e ${}_c\int f(z)dz = 2\pi i \frac{1}{4i}$

i.e ${}_{c_R}\int f(z)dz + \int_{-R}^{R} f(x)dx = \frac{\pi}{2}$

Taking limit as $R\to\infty$ on both sides, we get

$$\lim_{R\to\infty} {}_{c_R}\int f(z)dz + \lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \lim_{R\to\infty}\frac{\pi}{2}$$

$$\lim_{R\to\infty} {}_{c_R}\int f(z)dz + \int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{2}$$ **(b'cz limit of constant is constant)**

$$\therefore \quad \int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{2} - \lim_{R\to\infty} {}_{c_R}\int f(z)dz$$

i. e $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}\ dx = \frac{\pi}{2} - \lim_{R\to\infty} {}_{c_R}\int \frac{dx}{(1+z^2)^2}\ dz$ **----------------------(1)**

Consider $\left|{}_{c_R}\int \frac{1}{(1+z^2)^2}\ dz\right| \le {}_{c_R}\int \left|\frac{1}{(1+z^2)^2}\right| |dz| = {}_{c_R}\int \frac{1}{|(z^2+1)^2|}\ |dz|$

$\le {}_{c_R}\int \frac{1}{(|z|^2-1)^2}\ |dz|$ **[b'cz** $\frac{1}{|a+b|} \le \frac{1}{|a|-|b|}$ **]**

$= {}_{c_R}\int \frac{1}{(R^2-1)^2}\ |dz|$ **[b'cz [z] = R]**

$= \frac{1}{(R^2-1)^2}\ {}_{c_R}\int |dz|$

$= \frac{1}{(R^2-1)^2}$ **length of the semi circle C_R**

$= \frac{1}{(R^2-1)^2}$ **(π R) → 0 as R → ∞**

Thus $\lim_{R\to\infty}\left|{}_{c_R}\int \frac{1}{(1+z^2)^2} dz\right| = 0$

=> $\lim_{R\to\infty} {}_{c_R}\int \frac{1}{(1+z^2)^2} dz = 0$

From (1) given integral becomes $\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\ dx = \frac{\pi}{2} - 0$

i. e 2 $\int_0^{\infty} \frac{1}{(1+x^2)^2}\ dx = \frac{\pi}{2}$ **=>** $\int_0^{\infty} \frac{1}{(1+x^2)^2}\ dx = \frac{\pi}{4}$

$$\therefore \int_0^{\infty} \frac{1}{(1+x^2)^2}\ dx = \frac{\pi}{4}$$

Note: In example (2) power of D^r is 2, ∴ poles r of order 2 and hence according to that we have to calculate residues and proceed.

3. Prove by contour integration that ${}_0\int^{\infty} \frac{1}{(1+x^2)^3}$ **dx = 3π/8.**

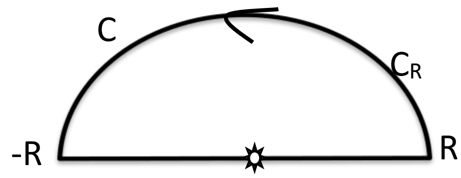
Soln: HOME work

Solve as above example, poles w get as order 3.

4. Evaluate by contour integration that $\int_0^\infty \frac{x^2\, dx}{(1+x^2)^2}$

Soln: Given integral $\int_0^\infty \frac{x^2\, dx}{(1+x^2)^2}$

Consider the integral $\int_C f(z)dz = \int_C \frac{z^2 dz}{(1+z^2)^2}$, taken around the closed contour C consisting of upper half large circle C_R : |z|=R and real line from –R to R



Here f(z) = $\frac{z^2}{(1+z^2)^2} = \frac{z^2}{((z+i)(z-i))^2} = \frac{z^2}{(z+i)^2\ (z-i)^2}$

Clearly Z=i, -i are poles of f(z) of order 2 for which Z = i lies inside C (where as z = -i = (0, -1) lies lower part of z-plane but our region is only upper half of z-plane)

$\therefore$ calculate residue at pole z=i (order is 2)

If R_1 is the residue of f(z) at z = i then R_2 = == $\frac{1}{(2-1)!} \lim_{z\to i} \frac{d}{dz}(z-i)^2 f(z)$

$$= \lim_{z\to i} \frac{d}{dz}(z-i)^2 \frac{z^2}{(z+i)^2\ (z-i)^2}$$

$$= \lim_{z\to i} \frac{d}{dz} \frac{z^2}{(z+i)^2} = \lim_{z\to i} \frac{2iz}{(z+i)^3} = \frac{-2}{(2i)^3} = \frac{-2}{-8i} = \frac{1}{4i}$$

By C.R. Theorem we have $\int_C f(z)dz$ **= 2πi (R_1)**

i.e $\int_C f(z)dz = 2\pi i \frac{1}{4i}$

i.e $\int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = \frac{\pi}{2}$

Taking limit as $R \to \infty$ **on both sides, we get**

$$\lim_{R\to\infty} \int_{C_R} f(z)dz + \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \lim_{R\to\infty} \frac{\pi}{2}$$

$$\lim_{R\to\infty} \int_{C_R} f(z)dz + \int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{2}$$ **(b'cz limit of constant is constant)**

$$\therefore \quad \int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{2} - \lim_{R\to\infty} \int_{C_R} f(z)dz$$

i.e $\int_{-\infty}^{\infty} \frac{x^2 dx}{(1+x^2)^2}\ dx = \frac{\pi}{2} - \lim_{R\to\infty} \int_{C_R} \frac{z^2\, dx}{(1+z^2)^2}\ dz$ **----------------------(1)**

Consider $\left| \int_{C_R} \frac{z^2}{(1+z^2)^2}\ dz \right| \le \int_{C_R} \left| \frac{z^2}{(1+z^2)^2} \right| |dz| = \int_{C_R} \frac{|z^2|}{|(z^2+1)^2|} |dz| = \int_{C_R} \frac{|z|^2}{|(z^2+1)^2|} |dz|$

$\le \int_{C_R} \frac{|z|^2}{(|z|^2-1)^2} |dz|$ **[b'cz** $\frac{1}{|a+b|} \le \frac{1}{|a|-|b|}$ **]**

$= \int_{C_R} \frac{R^2}{(R^2-1)^2} |dz|$ **[b'cz [z] = R]**

$= \frac{R^2}{(R^2-1)^2} \int_{C_R} |dz|$

$= \frac{R^2}{(R^2-1)^2}$ **length of the semi circle C_R**

$= \frac{R^2}{(R^2-1)^2}$ **(π R) → 0 as R → ∞ (b'cz degree of N^r< degree of D^r]**

Thus $\lim_{R\to\infty} \left|{}_{C_R}\int \frac{z^2}{(1+z^2)^2} dz\right| = 0$

=> $\lim_{R\to\infty} {}_{C_R}\int \frac{z^2}{(1+z^2)^2} dz = 0$

From (1) given integral becomes $\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2} \; dx = \frac{\pi}{2} - 0$

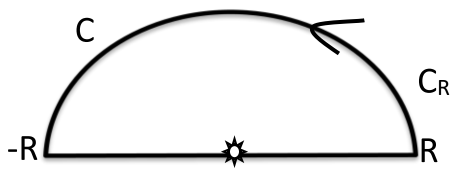
i. e $2\int_0^{\infty} \frac{x^2}{(1+x^2)^2} \; dx = \frac{\pi}{2}$ **=>** $\int_0^{\infty} \frac{x^2}{(1+x^2)^2} \; dx = \frac{\pi}{4}$

$\therefore \int_0^{\infty} \frac{x^2}{(1+x^2)^2} \; dx = \frac{\pi}{4}$

4. Prove by contour integration that ${}_{-\infty}\int^{\infty} \frac{x^2\,dx}{(a^2+x^2)^3}$**dx = π/8a³**, a>0

Soln: Given integral ${}_0\int^{\infty} \frac{x^2\,dx}{(a^2+x^2)^3}$

Consider the integral ${}_c\int f(z)dz = {}_c\int \frac{z^2 dz}{(a^2+z^2)^3}$, taken around the closed contour C consisting of upper half large circle C_R : |z|=R and real line from –R to R



Here $f(z) = \frac{z^2}{(a^2+z^2)^3} = \frac{z^2}{((z+ai)(z-ai))^3} = \frac{z^2}{(z+ai)^3\,(z-ai)^3}$

Clearly Z= ai, -ai are poles of f(z) of order 3 for which Z = ai lies inside C (where as z = -ai = (0, -a) (a>0 given), lies lower part of z-plane but our region is only upper half of z-plane)

$\therefore$calculate residue at pole z=ai (order is 3)

If R_1 is the residue of f(z) at z = ai then $R_1 = \frac{1}{(3-1)!} \lim_{z\to ai} \frac{d^2}{dz^2}(z-ai)^3 f(z)$

$$= \lim_{z\to ai} \frac{d^2}{dz^2}(z-ai)^2 \frac{z^2}{(z+ai)^3\,(z-ai)^3}$$

$$= \frac{1}{2!}\lim_{z\to ai} \frac{d^2}{dz^2}\frac{z^2}{(z+ai)^3} = \frac{1}{2}\lim_{z\to ai} \frac{d}{dz}\frac{2iaz-z^2}{(z+ai)^4}$$

$$= \frac{1}{2}\lim_{z\to ai} \frac{2z^2-2a^2-8iaz}{(z+ai)^5} = \frac{1}{2}\,\frac{4a^2}{32\,a^5\,i} = \frac{1}{16\,a^3\,i}$$

By C.R. Theorem we have ${}_c\int f(z)dz$ **= 2πi (R_1)**

i.e ${}_c\int f(z)dz$ **= 2πi** $\frac{1}{16\,a^3\,i}$

i.e ${}_{C_R}\int f(z)dz + \int_{-R}^{R} f(x)dx = \frac{\pi}{8\,a^3}$

Taking limit as $R\to\infty$ **on both sides, we get**

$$\lim_{R\to\infty} {}_{C_R}\int f(z)dz + \lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \lim_{R\to\infty} \frac{\pi}{8\,a^3}$$

$\lim_{R\to\infty} {}_{C_R}\int f(z)dz + \int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{8\,a^3}$ **(b'cz limit of constant is constant)**

$\therefore \int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{8\,a^3} - \lim_{R\to\infty} {}_{C_R}\int f(z)dz$

i.e $\int_{-\infty}^{\infty} \frac{x^2 dx}{(1+x^2)^3} \quad dx = \frac{\pi}{8\,a^3} - \lim_{R\to\infty} {}_{C_R}\int \frac{z^2\,dx}{(1+z^2)^3} \quad dz$ **----------------------(1)**

Consider $\left| {}_{C_R}\int \frac{z^2}{(1+z^2)^2} \quad dz \right| \le {}_{C_R}\int \left|\frac{z^2}{(1+z^2)^3}\right| \quad |dz| = {}_{C_R}\int \frac{|z^2|}{|(z^2+1)^3|} \quad |dz| = {}_{C_R}\int \frac{|z|^2}{|(z^2+1)^3|} \quad |dz|$

$\le {}_{C_R}\int \frac{|z|^2}{(|z|^2-1)^3} \quad |dz|$ **[b'cz** $\frac{1}{|a+b|} \le \frac{1}{|a|-|b|}$ **]**

$= {}_{C_R}\int \frac{R^2}{(R^2-1)^3} \quad |dz|$ **[b'cz lzl = R]**

$= \frac{R^2}{(R^2-1)^3} \, {}_{C_R}\int \quad |dz|$

$= \frac{R^2}{(R^2-1)^3}$ **length of the semi circle C_R**

$= \frac{R^2}{(R^2-1)^3}(\pi R) \to 0$ **as** $R \to \infty$ **(b'cz degree of N^r< degree of D^r]**

Thus $\lim_{R\to\infty} \left| {}_{C_R}\int \frac{z^2}{(1+z^2)^3} dz \right| = 0$

=> $\lim_{R\to\infty} {}_{C_R}\int \frac{z^2}{(1+z^2)^3} dz = 0$

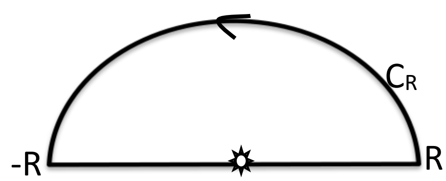
From (1) given integral becomes $\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^3} \quad dx = \frac{\pi}{8\,a^3} - 0 = \frac{\pi}{8\,a^3}$

$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2} \quad dx = \frac{\pi}{8\,a^3}$

5. Prove that $\int_0^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{12}$ **by contour integration. (2009, 2018)**

Soln: Given integral $\int_0^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$

Consider the integral ${}_C\int f(z)dz = {}_C\int \frac{dz}{(z^2+1)(z^2+4)}$, taken around the closed contour C consisting of upper half large circle C_R : lzl=R and real line from –R to R



Here f(z) = $\frac{1}{(z^2+1)(z^2+4)} = \frac{1}{(z+i)(z-i)(z+2i)(z-2i)}$

Clearly Z=i, -i, 2i, -2i are simple poles of f(z) for which Z = i and 2i lies inside C (where as z = -i i.e (0, -1) and z=-2i i.e (0, -2) lies in lower part of z-plane but our region is only upper half of z-plane)

$\therefore$ calculate residue at z=i and 2i

If R_1 is the residue of f(z) at z =i then $R_1 = \lim_{z\to i}(z-i)\,f(z)$

$$= \lim_{z\to i}(z-i)\ \frac{1}{(z+i)(z-i)(z+2i)(z-2i)}$$

$$= \lim_{z\to i}\ \frac{\mathbf{1}}{(z+i)(z+2i)(z-2i)}\ = \frac{\mathbf{1}}{\mathbf{2i(3)}}$$

$R_1 = \frac{\mathbf{1}}{\mathbf{6i}}$

If R_2 is the residue of f(z) at z =2i then R_1 = = $\lim_{z\to 2\,i}(z-2i)\ f(z)$

$$= \lim_{z\to 2i}(z-2i)\ \frac{1}{(z+i)(z-i)(z+2i)(z-2i)}$$

$$= \lim_{z\to 2i}\ \frac{\mathbf{1}}{(z+2i)(z+i)(z-i)}\ = \frac{\mathbf{1}}{\mathbf{4i(-3)}}\ = \frac{\mathbf{1}}{\mathbf{-12i}}$$

$R_2 = \frac{\mathbf{1}}{\mathbf{-12i}}$

By C.R. Theorem we have ${}_C\int f(z)dz = 2\pi i\ (R_1 + R_2)$

i.e ${}_C\int f(z)dz = 2\pi i(\frac{1}{6i} + \frac{1}{-12i})$

i.e ${}_{C_R}\int f(z)dz + \int_{-R}^{R} f(x)dx = \frac{2\pi i}{12i} = \frac{\pi}{6}$

Taking limit as $R \to \infty$ on both sides, we get

$$\lim_{R\to\infty} {}_{C_R}\int f(z)dz + \lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \lim_{R\to\infty}\frac{\pi}{6}$$

$$\lim_{R\to\infty} {}_{C_R}\int f(z)dz + \int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{6}$$ (b'cz limit of constant is constant)

$$\therefore \qquad \int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{6} - \lim_{R\to\infty} {}_{C_R}\int f(z)dz$$

i.e $\int_{-\infty}^{\infty}\frac{\mathbf{1}}{(x^2+1)(x^2+4)}\ dx = \frac{\pi}{6} - \lim_{R\to\infty} {}_{C_R}\int \frac{\mathbf{1}}{(z^2+1)(z^2+4)}\ dz$ ---------------------(1)

Consider $\left|{}_{C_R}\int \frac{\mathbf{1}}{(z^2+1)(z^2+4)}\ dz\right| \le {}_{C_R}\int \left|\frac{\mathbf{1}}{(z^2+1)(z^2+4)}\right|\ |dz| = {}_{C_R}\int \frac{1}{|z^2+1|}\ \frac{1}{|z^2+4|}|dz|$

$\le\ {}_{C_R}\int \frac{1}{(|z|^2-1)(|z|^2-4)}\ |dz|$ [b'cz $\frac{1}{|a+b|} \le \frac{1}{|a|-|b|}$]

$=\ {}_{C_R}\int \frac{1}{(R^2-1)R^2-4)}\ |dz|$ [b'cz [z] = R]

$= \frac{1}{(R^2-1)R^2-4)}\ {}_{C_R}\int\ |dz|$

$= \frac{1}{(R^2-1)R^2-4)}$ length of the semi circle C_R

$= \frac{1}{(R^2-1)R^2-4)}(\pi R) \to 0$ as $R \to \infty$

Thus $\lim_{R\to\infty}\left|{}_{C_R}\int \frac{\mathbf{1}}{(z^2+1)(z^2+4)}\ dz\right| = 0$

=> $\lim_{R\to\infty} {}_{C_R}\int \frac{\mathbf{1}}{(z^2+1)(z^2+4)}\ dz = 0$

From (1) given integral becomes $\int_{-\infty}^{\infty}\frac{\mathbf{1}}{(x^2+1)(x^2+4)}\ dx = \frac{\pi}{6} - 0 = \frac{\pi}{6}$

i. e $2\int_{0}^{\infty}\frac{\mathbf{1}}{(x^2+1)(x^2+4)}\ dx = \frac{\pi}{6}$ [b'cz $\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a}\frac{1}{1+x^2}\ dx\ \ as\ f(x) is\ even\ fuction$]

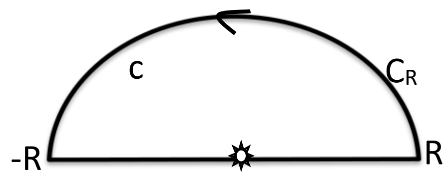
=> $\int_{0}^{\infty}\frac{\mathbf{1}}{(x^2+1)(x^2+4)}\ dx = \frac{\pi}{12}$

$$\therefore \int_0^{\infty} \frac{1}{(x^2+1)(x^2+4)} \quad dx = \frac{\pi}{12}$$

6. Prove that $\quad {}_0\int^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+4)^2} = \frac{\pi}{200}$ **by contour integration.**

Soln: Given integral ${}_0\int^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+4)^2}$

Consider the integral ${}_C\int f(z)dz = {}_C\int \frac{z^2 dz}{(z^2+9)(z^2+4)^2}$, taken around the closed contour C consisting of upper half large circle C_R : |z|=R and real line from –R to R



Here f(z) = $\frac{z^2}{(z^2+9)(z^2+4)^2} = \frac{z^2}{(z+3i)(z-3i)(z-2i)^2(z+2i)^2}$

Clearly Z=3i, -3i are simple poles and z= 2i, -2i are poles of order 2 of f(z) for which Z = i and 2i lies inside C (where as z = -3i i.e (0, -3) and z=-2i i.e (0, -2) lies in lower part of z-plane but our region is only upper half of z-plane)

∴calculate residue at z=3i and 2i

If R_1 is the residue of f(z) at z =3i then $R_1 = \lim_{z\to 3i}(z-3i)\, f(z)$ (b'cz z=3i is simple pole)

$$= \lim_{z\to 3i} (z-3i) \frac{z^2}{(z+3i)(z-3i)(z^2+4)^2}$$

$$= \lim_{z\to 3i} \frac{z^2}{(z+3i)(z^2+4)^2} = \frac{-9}{6i(-5)^2}$$

$$R_1 = \frac{-3}{50i}$$

If R_2 is the residue of f(z) at z =2i then $R_2 = \frac{1}{(2-1)!} \lim_{z\to 2i} \frac{d}{dz}(z-2i)^2 f(z)$

$$= \lim_{z\to 2i} \frac{d}{dz}(z-2i)^2 \frac{z^2}{(z+3i)(z-3i)(z-2i)^2(z+2i)^2}$$

$$= \lim_{z\to 2i} \frac{d}{dz} \frac{z^2}{(z+3i)(z-3i)(z+2i)^2}$$

$$= \lim_{z\to 2i} \frac{d}{dz} \left[\frac{z^2}{(z^2+9)(z+2i)^2}\right]$$

$$= \lim_{z\to 2i} \frac{4i\{[5)4-i(z^2+9)-z^2(z+2i)]}{(z^2+9)^2(z+2i)^3}$$

$$R_2 = \frac{4i[(5)(4i)-2i(5)-(-4)(4i)]}{(5)^2(4i)^3} = \frac{4i(26i)}{(25)(-64i)} =$$

$$= \frac{-13}{-200i} = \frac{13}{200i}$$

By C.R. Theorem we have ${}_C\int f(z)dz = 2\pi i\,(R_1 + R_2)$

i.e ${}_{c}\int f(z)dz = 2\pi i\left(\frac{-3}{50i} + \frac{13}{200i}\right) = 2\pi i\left(\frac{1}{200i}\right) = \frac{\pi}{100}$

i.e ${}_{C_R}\int f(z)dz + \int_{-R}^{R} f(x)dx = \frac{\pi}{100}$

Taking limit as $R \to \infty$ on both sides, we get

$\lim_{R\to\infty} {}_{C_R}\int f(z)dz + \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \lim_{R\to\infty} \frac{\pi}{100}$

$\lim_{R\to\infty} {}_{C_R}\int f(z)dz + \int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{100}$ (b'cz limit of constant is constant)

$\therefore \int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{100} - \lim_{R\to\infty} {}_{C_R}\int f(z)dz$

i.e $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2} dx = \frac{\pi}{100} - \lim_{R\to\infty} {}_{C_R}\int \frac{z^2}{(z^2+9)(z^2+4)^2} dz$ ----------------------(1)

Consider $\left| {}_{C_R}\int \frac{z^2}{(z^2+9)(z^2+4)^2} dz \right| \leq {}_{C_R}\int \left|\frac{z^2}{(z^2+9)(z^2+4)^2}\right| |dz| = {}_{C_R}\int \frac{|z|^2}{|z^2+9|} \frac{1}{|(z^2+4)|^2} |dz|$

$\leq {}_{C_R}\int \frac{|z|^2}{(|z|^2-9)\,(|z|^2-4)^2} |dz|$ [b'cz $\frac{1}{|a+b|} \leq \frac{1}{|a|-|b|}$]

$= {}_{C_R}\int \frac{R^2}{(R^2-9)\,(R^2-4)^2} |dz|$ [b'cz |z| = R]

$= \frac{R^2}{(R^2-9)\,(R^2-4)^2} {}_{C_R}\int |dz|$

$= \frac{R^2}{(R^2-9)\,(R^2-4)^2}$ length of the semi circle C_R

$= \frac{R^2}{(R^2-9)\,(R^2-4)^2} (\pi R) \to 0$ as $R \to \infty$ (degree of R in Nr $<$ *degree of R in D*r)

Thus $\lim_{R\to\infty} \left| {}_{C_R}\int \frac{z^2}{(z^2+9)(z^2+4)^2} dz \right| = 0$

=> $\lim_{R\to\infty} {}_{C_R}\int \frac{z^2}{(z^2+9)(z^2+4)^2} dz = 0$

From (1) given integral becomes $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2} dx = \frac{\pi}{100} - 0 = \frac{\pi}{100}$

i. e $2\int_{0}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2} dx = \frac{\pi}{100}$ [b'cz $\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} \frac{1}{1+x^2} dx$ *as f(x)is even fuction*]

=> $\int_{0}^{\infty} \frac{z^2}{(x^2+9)(x^2+4)^2} dx = \frac{\pi}{200}$

$\therefore \int_{0}^{\infty} \frac{z^2}{(x^2+9)(x^2+4)^2} dx = \frac{\pi}{200}$

Examples where N^r is in terms trigonometric function (These examples are also important)

7. Prove that ${}_0\int^{\infty} \frac{\cos ax}{(x^2+1)}$ dx = $\frac{\pi}{2e^a}$ by contour integration.(2012)

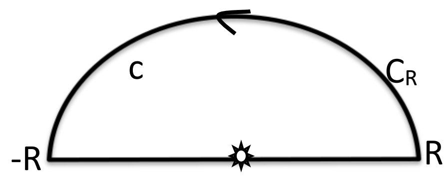
Soln: Given integral $\int_{0}^{\infty} \frac{\cos ax}{(x^2+1)} dx$ (starting procedure is little change)

We know that cosax = Real Part of (cosax + isinax) = R.P of e^{iax}

$\therefore \frac{\cos ax}{(x^2+1)}$ = R.P of $\frac{e^{iax}}{(x^2+1)}$

$\Rightarrow \int_0^\infty \frac{\boldsymbol{cosax}}{(\boldsymbol{x^2+1})} \quad dx$ = R.P of $\int_0^\infty \frac{\boldsymbol{e^{iax}}}{(\boldsymbol{x^2+1})} \quad dx$-----------------------(1)

Consider the integral Consider the integral $_C\int f(z)dz = {}_C\int \frac{e^{iaz}dz}{(z^2+1)}$, taken around the closed contour C consisting of upper half large circle C_R : lzl=R and real line from –R to R



Here f(z) = $\frac{e^{iaz}}{z^2+1} = \frac{e^{iaz}}{(z+i)(z-i)}$

Clearly Z=i, -i are simple poles of f(z) for which Z = i lies inside C (where as z = -i = (0, -1) lies lower part of z-plane but our region is only upper half of z-plane)

$\therefore$ calculate residue at z=i

If R_1 is the residue of f(z) at z =i then R_1 = = $\lim_{z\to i}(z-i)\, f(z)$

$= \lim_{z\to i}(z-i)\frac{e^{iaz}}{(z+i)(z-i)}$

$= \lim_{z\to i}\frac{e^{iaz}}{(z+i)} = \frac{e^{-a}}{2i}$

By C.R. Theorem we have $_C\int f(z)dz = 2\pi i\ (R_1)$

i.e $_C\int f(z)dz = 2\pi i\, \frac{e^{-a}}{2i}$

i.e $_{C_R}\int f(z)dz + \int_{-R}^{R} f(x)dx = \pi e^{-a}$

Taking limit as $R \to \infty$ on both sides, we get

$\lim_{R\to\infty} {}_{C_R}\int f(z)dz + \lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \lim_{R\to\infty} \pi\, e^{-a}$

$\lim_{R\to\infty} {}_{C_R}\int f(z)dz + \int_{-\infty}^{\infty} f(x)dx = \pi e^{-a}$ (b'cz limit of constant is constant)

$\therefore \int_{-\infty}^{\infty} f(x)dx = \pi e^{-a} - \lim_{R\to\infty} {}_{C_R}\int f(z)dz$

i. e $\int_{-\infty}^{\infty}\frac{e^{iax}}{1+x^2} \quad dx = \pi e^{-a} - \lim_{R\to\infty} {}_{C_R}\int \frac{e^{iaz}}{1+z^2} \quad dz$ ----------------------(2)

Consider $\lim_{R\to\infty} {}_{C_R}\int \frac{e^{iaz}}{1+z^2} \quad dz = \lim_{R\to\infty} {}_{C_R}\int e^{iaz}\frac{1}{1+z^2} \quad dz$

which is in the form of $\boldsymbol{\lim_{R\to\infty} {}_{C_R}\int e^{imz} f(z)dz}$ **where f(z) =** $\frac{1}{1+z^2}$ & m=a

and $\boldsymbol{\lim_{lzl\to\infty}|f(z)|} = \boldsymbol{\lim_{lzl\to\infty}}\frac{1}{1+z^2} = 0$

$\therefore$ **by Jordan's Lemma,** $\boldsymbol{\lim_{R\to\infty} {}_{C_R}\int e^{iaz}\frac{1}{1+z^2} \quad dz} = 0$ (in these examples we r using J.Lemm)

From (2) we have $\int_{-\infty}^{\infty}\frac{e^{iax}}{1+x^2} \quad dx = \pi e^{-a} - 0$

i. e $2\int_0^\infty \frac{e^{iax}}{1+x^2} \quad dx = \pi e^{-a}$ [b'cz $\int_{-a}^{a} f(x)\ dx = 2\int_0^a \frac{1}{1+x^2}\ dx\ as\ f(x) is\ even\ fuction$]

$\therefore \int_0^\infty \frac{e^{iax}}{1+x^2} \quad dx = ½\, \pi e^{-a}$

From (1) given integral $\int_0^\infty \frac{cosax}{1+x^2} \quad dx$ = R.P.of $\int_0^\infty \frac{e^{iax}}{1+x^2} \quad dx$ = R.P. of (½ πe^{-a})

= ½ πe^{-a}

= $\frac{\pi}{2e^a}$

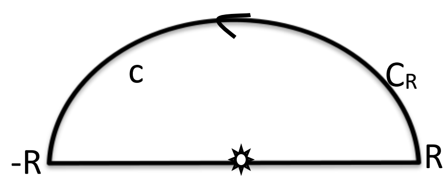
$\therefore \int_0^\infty \frac{cosax}{1+x^2} \quad dx = \frac{\pi}{2e^a}$

8. Prove that $_0\int^\infty$ cosmx dx /(a^2+x^2) = (π/2a)e^{-ma} , a,m>0 by contour integration.(2011, 13, 14)
HOME work: same as above example , in the place of x^2+1 it is x^2+a^2, so poles are ai, -ai.

8. Prove that $_0\int^\infty \frac{cosmx}{(a^2+x^2)^2}$ dx = $\frac{\pi}{4a^2}$ (1+ma)e^{-ma} , a,m>0 by contour integration.

Soln: Given integral $_0\int^\infty \frac{cosmx}{(a^2+x^2)^2}$ dx = R.P of $_0\int^\infty \frac{e^{imx}}{(a^2+x^2)^2}$ dx-------------(1)

Consider the integral Consider the integral $_c\int f(z)dz = {}_c\int \frac{e^{imz}dz}{(a^2+z^2)^2}$, taken around the closed contour C consisting of upper half large circle C_R : lzl=R and real line from –R to R



Here f(z) = $\frac{e^{imz}}{(a^2+z^2)^2} = \frac{e^{imz}}{[(z+ai)(z-ai)]^2} = \frac{e^{imz}}{(z+ai)^2(z-ai)^2}$

Clearly Z=ai, -ai are poles of order 2 of f(z) for which Z = ai lies inside C (where as z = -ai = (0, -a) lies lower part of z-plane (b'cz a>0) but our region is only upper half of z-plane)

$\therefore$calculate residue at z=ai

If R_1 is the residue of f(z) at z =ai then R_1 = = $\frac{1}{(2-1)!} \lim_{z\to ai} \frac{d}{dz}(z-ai)^2$ f(z)

$$= \lim_{z\to ai} \frac{d}{dz}(z-ai)^2 \frac{e^{imz}}{(z+ai)^2(z-ai)^2}$$

$$= \lim_{z\to ai} \frac{d}{dz} \frac{e^{imz}}{(z+ai)^2}$$

$$= \lim_{z\to ai} \left[\frac{(z+ai)^2\, e^{imz}(im) - e^{imz}2(z+ia)}{(z+ai)^4}\right]$$

$$= \lim_{z\to ai} \left[\frac{(z+ia)e^{imz}(im) - e^{imz}\ 2}{(z+ai)^3}\right]$$

$$R_1 = \frac{e^{-am}(2ia(im)-2)}{(2ai)^3} = \frac{-2e^{-am}(am+1)}{-8i(a)^3} = \frac{e^{-am}(am+1)}{4i(a)^3}$$

By C.R. Theorem we have $_c\int f(z)dz = 2\pi i\,(R_1)$

i.e $_c\int f(z)dz = 2\pi i\,\frac{e^{-am}(am+1)}{4i(a)^3}$

i.e $_{c_R}\int f(z)dz + \int_{-R}^{R} f(x)dx = \frac{\pi e^{-am}(am+1)}{2(a)^3}$

Taking limit as $R \to \infty$ on both sides, we get

$$\lim_{R\to\infty} {}_{c_R}\int f(z)dz + \lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \lim_{R\to\infty} \frac{\pi e^{-am}(am+1)}{2(a)^3}$$

$$\lim_{R\to\infty} {}_{c_R}\int f(z)dz + \int_{-\infty}^{\infty} f(x)dx = \frac{\pi e^{-am}(am+1)}{2(a)^3}$$ (b'cz limit of constant is constant)

$$\therefore \quad \int_{-\infty}^{\infty} f(x)dx = \frac{\pi e^{-am}(am+1)}{2(a)^3} - \lim_{R\to\infty} {}_{c_R}\int f(z)dz$$

i.e $\int_{-\infty}^{\infty} \frac{e^{imx}}{(a^2+x^2)^2}\, dx = \frac{\pi e^{-am}(am+1)}{2(a)^3} - \lim_{R\to\infty} {}_{c_R}\int \frac{e^{imz}}{(a^2+z^2)^2}\, dz$ ----------------------(2)

Consider $\lim_{R\to\infty} {}_{c_R}\int \frac{e^{imz}}{(a^2+z^2)^2}\, dz = \lim_{R\to\infty} {}_{c_R}\int e^{imz}\frac{1}{(a^2+z^2)^2}\, dz$

which is in the form of $\lim_{R\to\infty} {}_{c_R}\int e^{imz} f(z)dz$ **where f(z) =** $\frac{1}{(a^2+z^2)^2}$,

and $\lim_{|z|\to\infty} |f(z)| = \lim_{|z|\to\infty} \frac{1}{(a^2+z^2)^2} = 0$

$\therefore$ **by Jordan's Lemma,** $\lim_{R\to\infty} {}_{c_R}\int e^{imz} \frac{1}{(a^2+z^2)^2}\, dz = 0$ (in these examples we r using J.Lemm)

From (2) we have $\int_{-\infty}^{\infty} \frac{e^{imx}}{(a^2+x^2)^2}\, dx = \frac{\pi e^{-am}(am+1)}{2(a)^3} - 0$

i. e $2\int_0^{\infty} \frac{e^{imx}}{(a^2+x^2)^2}\, dx = \frac{\pi e^{-am}(am+1)}{2(a)^3}$ [b'cz $\int_{-a}^{a} f(x)\, dx = 2\int_0^{a} \frac{1}{(a^2+z^2)^2}\, dx$ $as\ f(x) is\ even\ fuction$]

$\therefore \int_0^{\infty} \frac{e^{imx}}{(a^2+x^2)^2}\, dx = \frac{\pi e^{-am}(am+1)}{4(a)^3}$

From (1) given integral $\int_0^{\infty} \frac{cosmx}{(a^2+x^2)^2}\, dx$ = R.P.of $\int_0^{\infty} \frac{e^{imx}}{(a^2+x^2)^2}\, dx$ = R.P. of $\left(\frac{\pi e^{-am}(am+1)}{4(a)^3}\right)$

$$= \frac{\pi e^{-am}(am+1)}{4(a)^3}$$

$$\therefore \int_0^{\infty} \frac{cosmx}{(a^2+x^2)^2}\, dx = \frac{\pi e^{-am}(am+1)}{4(a)^3}$$

In above example, particularly (i) if m=1 then $\int_0^{\infty} \frac{cosx}{(a^2+x^2)^2}\, dx = \frac{\pi e^{-a}(a+1)}{4(a)^3}$

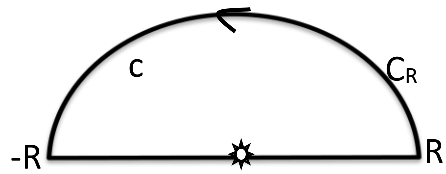
(ii) if a=1 then $\int_0^{\infty} \frac{cosmx}{(1+x^2)^2}\, dx = \frac{\pi e^{-m}(m+1)}{4}$

(iii) If m=1 and a=1 then $\int_0^{\infty} \frac{cosx}{(1+x^2)^2}\, dx = \frac{\pi e^{-1}2}{4} = \frac{\pi}{2e}$

9. Prove that $_0\int^{\infty} \frac{x\sin x}{(a^2+x^2)}$ **dx =** $\frac{\pi}{2e^a}$ **a,>0 by contour integration.**

Soln: Now integral $_{-\infty}\int^{\infty}\frac{x\sin x}{(a^2+x^2)}\ dx$ = Imaginary Part of $_{\infty}\int^{\infty}\frac{xe^{ix}}{(a^2+x^2)}\ dx$ [b'cz of sinx]

Consider the integral Consider the integral $_C\int f(z)dz$ = $_C\int \frac{ze^{iz}dz}{(a^2+z^2)}$, taken around the closed contour C consisting of upper half large circle C_R : IzI=R and real line from –R to R



Here f(z) = $\frac{ze^{iz}}{z^2+a^2}$ = $\frac{ze^{iz}}{(z+ai)(z-ai)}$

Clearly Z=ai, -ai are simple poles of f(z) for which Z = ai lies inside C (where as z = -ai = (0, -a) lies lower part of z-plane but our region is only upper half of z-plane)

∴calculate residue at z=ai

If R_1 is the residue of f(z) at z =ai then R_1 = = $\lim_{z\to i}(\mathrm{z}-\mathrm{ai})\ \mathrm{f(z)}$

$$= \lim_{z\to ai}(\mathrm{z}-\mathrm{ai})\frac{ze^{z}}{(z+ai)(z-ai)}$$

$$= \lim_{z\to ai}\frac{ze^{iz}}{(z+ai)} = \frac{ia\ e^{-a}}{2ia} = \frac{e^{-a}}{2}$$

By C.R. Theorem we have $_C\int f(z)dz = 2\pi i\ (R_1)$

i.e $_C\int f(z)dz = 2\pi i\,\frac{e^{-a}}{2} = i\pi e^{-a}$

i.e $_{C_R}\int f(z)dz\ + \int_{-R}^{R} f(x)dx\ = i\pi e^{-a}$

Taking limit as $R\to\infty$ on both sides, we get

$$\lim_{R\to\infty} {}_{C_R}\int f(z)dz\ + \lim_{R\to\infty}\int_{-R}^{R} f(x)dx\ = \lim_{R\to\infty} i\,\pi\, e^{-a}$$

$$\lim_{R\to\infty} {}_{C_R}\int f(z)dz\ + \int_{-\infty}^{\infty} f(x)dx\ = i\,\pi e^{-a}$$ (b'cz limit of constant is constant)

$$\therefore \quad \int_{-\infty}^{\infty} f(x)dx\ = i\pi e^{-a} - \lim_{R\to\infty} {}_{C_R}\int f(z)dz$$

i.e $\int_{-\infty}^{\infty}\frac{xe^{ix}}{x^2+a^2}\ dx\ = i\pi e^{-a} - \lim_{R\to\infty} {}_{C_R}\int \frac{ze^{iz}}{z^2+a^2}\ dz$ ---------------------(2)

Consider $\lim_{R\to\infty} {}_{C_R}\int \frac{ze^{iz}}{z^2+a^2}\ dz\ = \lim_{R\to\infty} {}_{C_R}\int e^{iz}\frac{z}{z^2+a^2}\ dz$

which is in the form of $\lim_{R\to\infty} {}_{C_R}\int e^{imz}f(z)dz$ **where f(z) =** $\frac{z}{z^2+a^2}$ & m=1

and $\lim_{|z|\to\infty}|f(z)|\ = \lim_{|z|\to\infty}\frac{z}{1+z^2}\ = 0$

∴ **by Jordan's Lemma,** $\lim_{R\to\infty} {}_{C_R}\int e^{iz}\frac{1}{1+z^2}\ dz\ = 0$

From (2) we have $\int_{-\infty}^{\infty}\frac{xe^{ix}}{x^2+a^2}\ dx\ = i\,\pi e^{-a} - 0$

$\therefore$ **integral** $\int_{-\infty}^{\infty} \frac{\boldsymbol{x\,sinx}}{1+x^2} \quad dx$ = I.P.of $\int_{-\infty}^{\infty} \frac{\boldsymbol{e^{ix}}}{1+x^2} \quad dx$ = I.P. of (i $\pi \boldsymbol{e^{-a}}$)

= $\pi \boldsymbol{e^{-a}}$ [b'cz imaginary part of imaginary no. is itself]

= $\boldsymbol{\frac{\pi}{e^a}}$

i. e 2 $\int_0^{\infty} \frac{\boldsymbol{xsinx}}{x^2+a^2} \quad dx$ = $\boldsymbol{\frac{\pi}{e^a}}$ [b'cz $\int_{-a}^{a} f(x)\ \ dx$ = 2$\int_0^{a} \frac{x\ sinx}{1+x^2}\ dx\ \ as\ f(x) is\ even\ fuction$]

$\therefore$ $\int_0^{\infty} \frac{\boldsymbol{xsinx}}{x^2+a^2} \quad dx$ = ½ $\pi \boldsymbol{e^{-a}}$

given integral $\int_0^{\infty} \frac{\boldsymbol{xsinx}}{x^2+a^2} \quad dx$ = $\boldsymbol{\frac{\pi}{2e^a}}$

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(iv)